INVARIANT MEASURES ON ABELIAN METRIC GROUPS

BY

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We consider complete countably additive σ-finite measures which vanish on singletons and are non-identically zero. A measure \( m \) defined on a σ-algebra \( \mathcal{M} \) of subsets of a group \((G, +)\) is called invariant iff \( g + A \in \mathcal{M} \) and \( m(g + A) = m(A) \) for any \( g \in G \) and \( A \in \mathcal{M} \). It is called symmetric iff \( -A \in \mathcal{M} \) and \( m(-A) = m(A) \) for any \( A \in \mathcal{M} \). If \( d \) is a metric on \((G, +)\), then \( d \) is called invariant iff \( d(x, y) = d(g + x, g + y) \) for any \( g, x, y \in G \). Given a metric \( d \) on \((G, +)\), a measure \( m \) is called \( d \)-invariant iff the following condition holds: for any subsets \( A, B \subseteq G \), whenever there exists a function \( f: A^{0100} \to B \) such that \( d(x, y) = d(f(x), f(y)) \) for any \( x, y \in A \), and the set \( A \) is \( m \)-measurable, then \( B \) is also \( m \)-measurable and \( m(A) = m(B) \). Since left translations are isometries for any invariant metric, a measure invariant with respect to an invariant metric is clearly invariant. The converse turns out to be sometimes true as well. Bandt [1] proved that every left Haar measure on a locally compact metric group is invariant with respect to every invariant metric. The aim of this paper is to show that this is a rather specific feature of Haar measures in the sense that many other invariant measures on groups do not enjoy it. Our main result is the following

**Theorem 1.** Let \((G, +)\) be an abelian group without elements of order 2. Then every invariant measure on \( G \) can be extended to an invariant measure which is not invariant with respect to any invariant metric.

Since for abelian groups the inverse element operation is an isometry for every invariant metric, Theorem 1 follows immediately from

**Theorem 2.** Let \((G, +)\) be an abelian group without elements of order 2. Then every invariant measure on \( G \) can be extended to an invariant non-symmetric measure.

**Proof.** We follow the idea from Pelc [2]. The crucial points of the argument are recalled in order to make the present paper self-contained. First we need the following lemma, essentially due to Szpiłrajn [3]. We omit the easy proof.
LEMMA A. Let $m$ be an invariant measure defined on a $\sigma$-algebra $\mathcal{M}$ of subsets of the group $(G, +)$ and let $E$ be a subset of $G$. If the family

$$I = \{X \in G : X \subset K + E \text{ for some countable subset } K \text{ of } G\}$$

consists of sets of inner measure zero, then $m$ can be extended to an invariant measure $\tilde{m}$ defined on the $\sigma$-algebra generated by $\mathcal{M} \cup I$ and vanishing on all sets from $I$.

It is easy to see that the only sets on which the extended measure $\tilde{m}$ assumes value zero are those of the form $X \cup Y$, where $m(X) = 0$ and $Y \in I$.

If the set $E$ from Lemma A is of positive outer measure and for any countable $K \subset G$ we have $m(-E \cap (K + E)) = 0$, then either $m(-E) = 0$ and the measure $m$ itself is non-symmetric or $-E$ has positive outer measure as well, and hence it is not of the form $X \cup Y$, where $m(X) = 0$ and $Y \in I$. This proves the following

LEMMA B. Let $m$ be an invariant measure defined on a $\sigma$-algebra $\mathcal{M}$ of subsets of the group $(G, +)$ and let $E$ be a subset of $G$ satisfying the following conditions:

(a) $E$ is a set of positive outer measure,
(b) for any countable $K \subset G$ the set $K + E$ has inner measure zero,
(c) for any countable $K \subset G$ the set $-E \cap (K + E)$ has measure zero.

Then $m$ can be extended to an invariant non-symmetric measure.

In order to prove our theorem it is enough to construct a set $E \subset G$ satisfying conditions (a), (b) and (c) of Lemma B. We shall call such a set a $(G, m)$-extender.

Case 1. Additive group of a linear space over a countable field.

Let $V$ be a linear space over a countable field $F$ and $m$ any measure on $V$ invariant with respect to addition. Fix a linear basis $B = \{v_\alpha : \alpha < \kappa\}$ of $V$ over $F$. For any positive integer $n$, let $V_n$ denote the set of those elements of $V$ which have $n$ summands in the basis $B$ representation. For any sequence $s = (f_1, \ldots, f_r)$ of elements of $F$ let $V_s$ denote the set of all elements of the form $f_1 v_{\alpha_1} + \ldots + f_r v_{\alpha_r}$ for some $\alpha_1 > \ldots > \alpha_r$. Let $n_0$ be the least natural number for which $V_{n_0}$ has positive outer measure and let $s_0$ be such a sequence $(f_1, \ldots, f_{n_0})$ for which $V_{s_0}$ has positive outer measure. We claim that $E = V_{s_0}$ is a $(V, m)$-extender. Condition (a) of Lemma B is trivially satisfied. In order to prove condition (b) let $K$ be any countable subset of $V$, and $H$ any uncountable subset of $B$ none of whose elements appears in the representation of any $k \in K$. It is easy to see that for distinct $h_1, h_2 \in H$ we have

$$(h_1 + K + E) \cap (h_2 + K + E) \subset K + h_2 + F \cdot h_1 + V_{n_0-1}.$$  

In view of the definition of $n_0$ and of the invariance of measure $m$ it follows that the set $K + E$ has inner measure zero.
Finally, we prove that $E$ satisfies condition (c) of Lemma B. Indeed, assume that for some countable set $K \subset G$ we have

$$x \in -E \cap (K+E).$$

Hence

\[ (*) \quad x = -f_1 v_{a_1} - \ldots - f_{n_0} v_{a_{n_0}} = k + f_1 v_{b_1} + \ldots + f_{n_0} v_{b_{n_0}}. \]

Let $D$ be the countable set consisting of those elements of $B$ which appear in the representation of some $k \in K$. It follows from $(*)$ that, for some $i$ $(1 \leq i \leq n_0)$, $v_{b_i} \in D$ (otherwise, $k = 0$, which contradicts the fact that $V$ does not have elements of order 2). Hence

$$-E \cap (K+E) \subset K + F \cdot D + V_{n_0-1}.$$  

However, the latter set is a union of countably many translates of $V_{n_0-1}$ (which has measure zero by definition) and, consequently, has measure zero. This completes the proof in case 1.

Case 2. Torsion free abelian group.

Let $G$ be a torsion free abelian group. There exists a homomorphic embedding of $G$ into the additive group of a linear space $V$ over the field $Q$ of rationals, such that a certain basis $B = \{v_x : x < x\}$ of $V$ consists of elements of $G$. Let $m$ be any invariant measure on $G$ and denote by $\tilde{m}$ its trivial extension to $V$ (putting 0 outside of $G$). Clearly, the measure $\tilde{m}$ is invariant with respect to translations from $G$.

Let $V_n$ and $V_s$ have the same meaning as in case 1. Denote by $n_0$ the least index $n$ for which $b + V_n$ has positive outer $\tilde{m}$-measure for some $b \in V$. Let $s_0$ be such a sequence $(q_1, \ldots, q_{n_0})$ of rationals for which $b + V_{s_0}$ has positive outer $\tilde{m}$-measure. Finally, put $E = (b + V_{s_0}) \cap G$. We claim that the set $E$ is a $(G, m)$-extender. Condition (a) of Lemma B is trivially satisfied. In order to prove condition (b) let $K$ be any countable subset of $G$. Take any uncountable set $H$ of elements of the basis $B$ which do not appear in the representation of any $k \in K$ or $b$. Let $c$ be a natural number distinct from all $\pm q_i, \pm q_i \pm q_j$ $(i, j \leq n_0)$. Clearly,

$$\left( c h_1 + K + E \right) \cap \left( c h_2 + K + E \right) = \emptyset$$

for distinct $h_1, h_2 \in H$ and $c h \in G$ for any $h \in H$. This proves that the set $K + E$ has inner measure zero.

It remains to prove condition (c) of Lemma B. Take any countable subset $K$ of the group $G$ and consider any element $x \in -E \cap (K+E)$. Hence

$$x = -b - q_1 v_{a_1} - \ldots - q_{n_0} v_{a_{n_0}} = k + b + q_1 v_{b_1} + \ldots + q_{n_0} v_{b_{n_0}}$$

for some $k \in K$. This implies

$$-k - 2b = q_1 v_{a_1} + \ldots + q_{n_0} v_{a_{n_0}} + q_1 v_{b_1} + \ldots + q_{n_0} v_{b_{n_0}}.$$
Denote by $D$ the countable subset of the basis $B$ consisting of all elements appearing in the representation of $b$ or of some $k \in K$. Hence $v_{h_i} \in D$ for some $i (1 \leq i \leq n_0)$. It follows that

$$-E \cap (K + E) \subseteq K + b + Q \cdot D + V_{n_0 - 1}.$$ 

By the definition of $n_0$ we get $\tilde{m}(K + b + Q \cdot D + V_{n_0 - 1}) = 0$, which implies $m(-E \cap (K + E)) = 0$ and completes the proof in case 2.

Case 3. The general case.

Let $(G, +)$ be an arbitrary abelian group without elements of order 2. Denote by $H$ its torsion subgroup.

If $m(H) = 0$, then we define a measure $m_1$ on the quotient group $G/H$ by putting

$$m_1(\{a + H : a \in A\}) = m(A + H)$$

for all sets $A$ such that $A + H$ is $m$-measurable.

The group $G/H$ is torsion free and the measure $m_1$ is invariant. Hence, in view of case 2, there exists a $(G/H, m_1)$-extender. Call it $E_1$. It is easy to see that the set $E = \bigcup E_1$ is a $(G, m)$-extender.

If the subgroup $H$ has positive outer measure, consider its subgroups $H_i$ consisting of elements whose orders divide $i$. Since

$$H = \bigcup_{i=1}^{\infty} H_i,$$

one of the groups $H_i$ must have positive outer measure. Let $i_0$ be the least index of such a group. We proceed by induction on the number $k$ of prime divisors of $i_0$ (counting multiple divisors many times). We may assume that $i_0$ is odd.

If $k = 1$, then $i_0$ is prime, and hence $H_{i_0}$ is the additive group of a linear space over the field $Z_{i_0}$. Let $E$ have the same meaning as in case 1 (for the linear space $H_{i_0}$ and the measure $m$). It is easy to see that $E$ satisfies condition (b) of Lemma B for $G$ and $m$. In order to show that $E$ is a $(G, m)$-extender it is enough to prove that

**Claim. For any countable set $K \subseteq G$, $m(-E \cap (K + E)) = 0$.**

Let $K$ be any countable subset of $G$ and let $K' = K \cap H_{i_0}$. In view of the argument in case 1 we get $m(-E \cap (K' + E)) = 0$. On the other hand, $-E \cap ((K \setminus K') + E) = \emptyset$. This proves the claim and completes the proof for $k = 1$.

Suppose that for $i_0$ having $k$ prime divisors there exists a $(G, m)$-extender $E \subseteq H_{i_0}$. Let now $i_0 = p_1 \cdots p_{k+1}$ ($p_i$ are primes, $k \geq 1$) and $H'$ be the subgroup of $H_{i_0}$ consisting of elements of order $p_1$.

Since $m(H') = 0$, we can define an invariant measure $m'$ on $G/H'$ just as before. $H_{i_0}/H'$ is a subgroup of $G/H'$ all of whose elements have orders
dividing the number $p_2 \cdots p_{k+1}$, and hence different from 2. By definition, $H_{i_0}/H'$ has positive outer $m'$-measure. Hence by the inductive hypothesis there exists a $(G/H', m')$-extender $E' \subset H_{i_0}/H'$. It is easy to see that the set $E = \bigcup E'$ is a $(G, m)$-extender. This completes the proof in the general case.

Our next result shows that Theorem 1 fails to be true for arbitrary abelian groups. We give an example of an abelian group $(G, +)$ and an invariant metric $d$ on $G$ such that, whenever subsets $A$ and $B$ are isometric, one must be a translation of the other. Hence every invariant measure on $G$ must be $d$-invariant. In our example every element of $G$ has order 2.

**Proposition 1.** There exist an abelian group $(G, +)$ and an invariant metric $d$ on $G$ such that every invariant measure on $G$ is $d$-invariant.

**Proof.** Take as $(G, +)$ the group of all sets of natural numbers with symmetric difference as the group operation. The metric $d$ is defined as follows: if $a, b \in \omega$, let $f: \omega \to \{0, 1\}$ be the characteristic function of $a\Delta b$. Define

$$d(a, b) = \sum_{i=0}^{\infty} \frac{f(i)}{3^i}.$$ 

It is easy to show that $d$ is actually an invariant metric on $(G, +)$.

**Claim.** For any $a \in G$ and $r \in R$ there exists at most one $b \in G$ such that $d(a, b) = r$.

Indeed, suppose that $d(a, b') = d(a, b'')$ and let $f$ and $g$ be the characteristic functions of $a\Delta b'$ and $a\Delta b''$, respectively. Since

$$\sum_{i=0}^{\infty} \frac{f(i)}{3^i} = \sum_{i=0}^{\infty} \frac{g(i)}{3^i},$$

it follows that $f(i) = g(i)$ for every natural $i$, and hence $a\Delta b' = a\Delta b''$, which gives $b' = b''$ and proves the claim.

Suppose that for some subsets $A$ and $B$ of our group $G$ there is an isometry $f: A \rightarrow B$ with respect to the metric $d$. Let $x$ be any element of $A$. Put $g = f(x) - x$. Then for any $a \in A$ we have

$$d(x, a) = d(f(x), f(a)) \quad \text{and} \quad d(x, a) = d(g + x, g + a).$$

Since $f(x) = g + x$, this implies $d(f(x), f(a)) = d(f(x), g + a)$, and hence, by the claim, we get $f(a) = g + a$ for any $a \in A$. This shows that $B$ is a translation of $A$. Hence, whenever $m$ is an invariant measure on $G$, it must be $d$-invariant as well.

We close the paper with some remarks on the existence of invariant non-symmetric measures on groups which are not abelian. Clearly, we have to assume that uncountably many elements do not have order 2; otherwise, every measure is symmetric. We do not know if Theorem 2 holds without
the assumption about commutativity of the group. We are even not able to
decide if every group without elements of order 2 carries an invariant non-
symmetric measure. Our last proposition shows that this is true under a
stronger assumption.

**Proposition 2.** Let \( G \) be an uncountable group for which the function
\( f: G \to G \) given by \( f(g) = 2g \) is one-to-one. Then \( G \) carries an invariant non-
symmetric measure.

**Proof.** Let \( (G, +) \) be as above. In order to prove the proposition it is
enough to construct a set \( A \subset G \) such that \(-A \neq K + A\) for any countable
\( K \subset G \). Then the desired measure \( m \) can be defined as follows: \( m(X) = 0 \) for
\( X \subset K + A \), where \( K \) is any countable subset of \( G \), and \( m(Y) = 1 \) for \( Y \) being
a complement of \( X \) as above.

Let \( \{g_\alpha: \alpha < \kappa\} \) be a one-to-one enumeration of \( G \). We define the
required set \( A = \{a_\alpha: \alpha < \kappa\} \) by induction. If \( \{a_\alpha: \alpha < \beta\} \) are already defined,
let \( a_\beta \) be an element outside of the group generated by \( \{g_\alpha: \alpha < \beta\} \cup \{a_\alpha: \alpha < \beta\} \)
such that \( 2a_\beta \) is outside of this group as well. The existence of an
element with this property follows from the assumptions about \( G \).

First assume that the cardinal \( \kappa \) has uncountable cofinality.

Suppose that \( K = \{g_{s_n}: n \in \omega\} \) is a countable subset of \( G \) and let \( \alpha
= \sup \{\alpha_n: n \in \omega\} \). It is enough to show that \(-a_\alpha \neq K + A\). Assume the
contrary. Then \(-a_\alpha = g_{s_n} + a_\beta \) for some natural number \( n \) and \( \beta < \kappa \). If
\( \beta > \alpha \), this implies that \( a_\beta \) belongs to the group generated by \( a_\alpha \) and \( g_{s_n} \). If \( \beta
< \alpha \), this implies that \( a_\alpha \) belongs to the group generated by \( a_\beta \) and \( g_{s_n} \). If \( \beta
= \alpha \), this implies the equality \( 2a_\alpha = -g_{s_n} \). In each case we get a contradiction
with the definition of \( A \). If \( cf(\kappa) = \omega \), let \( G_1 \) be any subgroup of \( G \) of
cardinality \( \omega \). We construct a set \( A \) for \( G_1 \) as above. For any countable
\( M \subset G \) we have

\[
M + A = [(M \cap G_1) + A] \cup [(M \setminus G_1) + A].
\]

Since \(-A\) is not contained in the first summand (by the above argument)
and the second summand is disjoint from \( G_1 \), we get \(-A \neq M + A\). This
completes the proof.

Let us finally remark that using the set \( A \) constructed above it is
possible to define an invariant measure enjoying a stronger property. If we put

\[
m_1(X) = 0
\]

for \( X \subset (K + A) \cup G \setminus (L - A) \), where \( K \) and \( L \) are countable subsets of \( G \), and

\[
m_1(Y) = 1
\]

for \( Y \) being a complement of \( X \) as above, we see that \( m_1 \) is an invariant non-
symmetric measure defined on a symmetric \( \sigma \)-algebra. Hence not only
is \( m_1 \) non-symmetric but also it does not have any symmetric extension.
We do not know if Theorem 2 can be improved in a similar way.
Acknowledgements. Thanks are due to Piotr Zakrzewski for his substantial criticism concerning the first version of this paper and to Janusz Pawlikowski for pointing out some errors.

REFERENCES


Reçu par la Rédaction le 6.9.1983;
en version modifiée le 30.9.1985