On density concomitants of the covariant curvature tensor in the two- and three-dimensional Riemann space

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1. Introduction. One of the basic questions of the theory of geometric objects is to determine the algebraic concomitants of a certain type for a given object.

In the case $n = 2$ the scalar concomitants of the mixed tensor have been determined by Golab [8] under the supposition that the functions in question are of class $C^1$. All the scalar concomitants of the mixed tensor in the $n$-dimensional space have been determined by Aczel and Hosszú [2], while those of the twice covariant tensor in the $n$-dimensional space were determined by Zajtz in [14].

By means of the analytic method Bieszk [3] has determined for the curvature tensor concomitants being either densities or tensors of second order in the two-dimensional space and linear concomitants being two-times covariant tensors in the three-dimensional space.

The same analytic method, reducing the system of functional equations to the system of differential equations of the first order, has been applied by Bieszk and Wegrzynowski in [5] and [6] to determine densities and vector concomitants of the antisymmetric tensor and linear concomitants of a tensor $T_{αβ}$ in the two-dimensional space.

Scalar concomitants of a tensor $T_{αβ}$ without regularity assumptions in the two-dimensional space were determined by Wegrzynowski in paper [13].

A certain general and uniform method reducing the determination of the concomitants of geometric objects to the question of determining certain special subgroups of the general linear group $GL_n$ was given by Zajtz and Siwek in [12].

In this paper we shall determine by the analytic method all density (scalar) concomitants of the covariant curvature tensor in the two- and three-dimensional Riemann space.

Another method of solving the above-mentioned problem will be given in a forthcoming paper by S. Topa.
2. Density (scalar) concomitants in the two-dimensional space $V_2$.

First of all we give some general notations. If the passage from one allowable coordinate system ($\lambda$) to another ($\lambda'$) is given by the system of functions

\[ \xi^\mu = \varphi^\mu(\xi^\lambda), \quad \lambda = 1, 2, \ldots, n, \quad \lambda' = 1', 2', \ldots, n', \]

where

\[ A^\lambda_\mu = \frac{\partial \varphi^\mu(\xi^\lambda)}{\partial \xi^\lambda}, \]

\[ J \overset{\text{df}}{=} \det(A^\lambda_\mu) \neq 0, \]

then for the inverse transformation

\[ \xi^\lambda = \psi^\lambda(\xi^\mu), \]

we introduce the notation

\[ A^\mu_\lambda = \frac{\partial \psi^\lambda(\xi^\mu)}{\partial \xi^\mu}. \]

Between $A^\lambda_\mu$ and $A^\mu_\lambda$ the following relations occur:

\[ A^\lambda_\mu = \frac{\text{minor } A^\mu_\lambda}{J}. \]

In a Riemann space $V_n$ the induced connexion is given by means of the Christoffel symbols of second kind

\[ \{_{\alpha \beta}^\gamma\} \overset{\text{df}}{=} \frac{1}{2} g^{\nu\sigma}(\partial_\alpha g_{\beta \nu} + \partial_\beta g_{\alpha \nu} - \partial_\gamma g_{\alpha \beta}), \]

where $g^{\alpha \beta}$ is the inverse tensor to the metric tensor $g_{\alpha \beta}$.

The curvature tensor (Riemann–Christoffel tensor) is defined as follows:

\[ R^\gamma_{\alpha \beta \nu} \overset{\text{df}}{=} 2\partial_{[\alpha} \{_{\beta \nu]}^\gamma + 2\{_{[\alpha |\nu]}^\gamma \{_{\beta]}^\sigma\}. \]

The so-called covariant curvature tensor is defined by

\[ R^\gamma_{\alpha \beta \nu \delta} \overset{\text{df}}{=} R^\gamma_{\alpha \beta \nu} g_{\delta \gamma}. \]

Tensor (9) has the well-known properties

\[ 1^\circ \quad R^\gamma_{\alpha \beta \nu \delta} = R^\nu_{\alpha \beta \delta}, \]

\[ 2^\circ \quad R^\gamma_{\alpha \beta \nu \delta} = -R^\nu_{\delta \beta \alpha} = -R^\alpha_{\beta \delta \nu}. \]

The number $N$ of the so-called essential components of the tensor $R^\gamma_{\alpha \beta \nu \delta}$ in space $V_n$ is defined by the formula

\[ N = \frac{n^2(n^2 - 1)}{12}, \]

(see [11], 90.9).
After having passed to the new coordinate system \((\lambda')\) the tensor \(R_{\alpha\beta\gamma\delta}\) has the components \(R_{\alpha'\beta'\gamma'\delta'}\) connected with \(R_{\alpha\beta\gamma\delta}\) by the formula
\[
R_{\alpha'\beta'\gamma'\delta'} = A_{\alpha'}^\alpha A_{\beta'}^\beta A_{\gamma'}^\gamma A_{\delta'}^\delta R_{\alpha\beta\gamma\delta}.
\]
(12)

Let us proceed to determine the density concomitants of \(R_{\alpha\beta\gamma\delta}\) in the two-dimensional Riemann space.

For \(n = 2, N = 1\), i.e. \(R_{\alpha\beta\gamma\delta}\) has only one essential component
\[
x = R_{1212}.
\]
(13)

For the matrix \(A_{\alpha'}^\alpha\), we introduce the shorter notation
\[
\begin{bmatrix}
p_1 & p_2 \\
p_3 & p_4
\end{bmatrix}
\overset{dt}{\to}
\begin{bmatrix}
A_{1'}^{1} & A_{1'}^{2} \\
A_{2'}^{1} & A_{2'}^{2}
\end{bmatrix}.
\]
(14)

Hence we have
\[
J = (p_1 p_4 - p_2 p_3)^{-1},
\]
and according to [11] (formula 92.1)
\[
x' = R_{1'2'12'} = (p_1 p_4 - p_2 p_3)^2 x,
\]
or more briefly
\[
x' = J^{-2} x.
\]
(15)

We seek an algebraic concomitant \(H\) of \(R_{\alpha\beta\gamma\delta}\), which is a density of weight \((-r)\).

For \(n = 2\), \(H\) is a function of \(x\) fulfilling the equation
\[
H(x') = \varepsilon |J|^r H(x)
\]
or
\[
H[(p_1 p_4 - p_2 p_3)^2 x] = \varepsilon |J|^r H(x),
\]
where
\[
\varepsilon = \begin{cases} 
1 & \text{for a Weyl density}, \\
\text{sgn} J & \text{for an ordinary density}.
\end{cases}
\]
(19)

Assuming that \(H(x)\) is of class \(C^1\), we reduce equation (17) to an ordinary differential one. We differentiate (18) with respect to \(p_1, p_2, p_3, p_4\) and next substitute
\[
p_1 = p_4 = 1, \quad p_2 = p_3 = 0.
\]
(20)

After this operation we get four equations reducing to
\[
2x H'(x) = -r H(x),
\]
in which there is no intervention of \(\varepsilon\).

We have to distinguish two cases, I and II.
I. Let us assume that \( r = 0 \).

Ia. If \( x = 0 \), the problem is trivial, because \( R_{1213} = 0 \) in every coordinate system.

Ib. If \( x \neq 0 \), then \( H'(x) = 0 \); hence \( H(x) = \text{const} \) and in this case only arbitrary scalars could be concomitants of the tensor \( R_{1213} \).

II. Let us assume that \( r \neq 0 \).

IIa. If \( x = 0 \), then we have again the trivial case.

IIb. If \( x \neq 0 \), then the general solution of (21) has the form

\[
H = \begin{cases} 
C_1 |x|^{-r/2} & \text{for } x > 0, \\
C_2 |x|^{-r/2} & \text{for } x < 0, 
\end{cases}
\]

where \( C_1, C_2 \) are arbitrary constants different from zero (different or equal).

It is easy to prove that if \( \varepsilon = 1 \), \( H(x) \) defined by (22) fulfills (17), while for \( \varepsilon = \text{sgn} x \), formula (22) is no solution of (17) in the whole domain but for \( x > 0 \) only.

Now we can state

**Theorem 1.** In the space \( V_2 \) the only scalar concomitants (of the class \( C^1 \)) of the tensor \( R_{abyc} \) are arbitrary scalars, while the only density concomitants of the weight \(-r\) are Weyl-densities of the form

\[
H(x) = C |x|^{-r/2},
\]

where the arbitrary constant \( C \) different from zero is given by

\[
C = \begin{cases} 
C_1 & \text{for } x > 0, \\
C_2 & \text{for } x < 0.
\end{cases}
\]

3. Density (scalar) concomitants in the Riemann space \( V_3 \). In accordance with (11) (Section 2) we have for \( n = 3, N = 6 \). We introduce the following shorter notations for the six essential components of the tensor \( R_{abyc} \):

\[
\begin{align*}
x_1 &= R_{1212}, & x_2 &= R_{1313}, & x_3 &= R_{2323}, \\
x_4 &= R_{1213}, & x_5 &= E_{1223}, & x_6 &= R_{1323}
\end{align*}
\]

and the shorter ones for the elements of the matrix \( A^j_i \):

\[
[a_{ij}] = \begin{bmatrix} A^j_i \end{bmatrix}, \quad \text{where } i, j = 1, 2, 3;
\]

\[
J = \det(a_{ij}) \neq 0.
\]

In accordance with (6) (Section 2) and (2) we have

\[
[A^j_i] = J^{-1} \begin{bmatrix} a_{12}a_{53} - a_{23}a_{22} & a_{23}a_{21} - a_{21}a_{33} & a_{21}a_{32} - a_{22}a_{31} \\
a_{13}a_{43} - a_{12}a_{33} & a_{11}a_{33} - a_{13}a_{31} & a_{12}a_{31} - a_{11}a_{32} \\
a_{12}a_{33} - a_{13}a_{22} & a_{13}a_{21} - a_{11}a_{23} & a_{11}a_{22} - a_{12}a_{21}
\end{bmatrix}.
\]
In the new coordinate system \((\lambda')\) the coordinates \(x'_{i} (i = 1, 2, 3, 4, 5, 6)\) of the tensor \(R_{\alpha\beta\gamma\delta}\) have the following form:

\[
\begin{align*}
x'_{1} &= J^{-2}[a_{33}^{2} x_{1} + a_{23}^{2} x_{2} + a_{13}^{2} x_{3} - 2a_{23} a_{33} x_{4} + 2a_{13} a_{33} x_{5} - 2a_{13} a_{23} x_{6}], \\
x'_{2} &= J^{-2}[a_{31}^{2} x_{1} + a_{21}^{2} x_{2} + a_{11}^{2} x_{3} - 2a_{21} a_{31} x_{4} + 2a_{11} a_{31} x_{5} - 2a_{11} a_{21} x_{6}], \\
x'_{3} &= J^{-2}[a_{31}^{2} x_{1} + a_{21}^{2} x_{2} + a_{11}^{2} x_{3} - 2a_{21} a_{31} x_{4} + 2a_{11} a_{31} x_{5} - 2a_{11} a_{21} x_{6}], \\
x'_{4} &= J^{-2}[-a_{32} a_{33} x_{1} - a_{22} a_{33} x_{2} - a_{12} a_{13} x_{3} + (a_{22} a_{33} + a_{23} a_{32}) x_{4}], \\
x'_{5} &= J^{-2}[-a_{12} a_{33} + a_{13} a_{32}) x_{5} + (a_{12} a_{23} + a_{13} a_{22}) x_{6}], \\
x'_{6} &= J^{-2}[-a_{12} a_{33} + a_{13} a_{32}) x_{5} + (a_{12} a_{23} + a_{13} a_{22}) x_{6}],
\end{align*}
\]

(5)

Similarly to Section 2 the sought density \(H\) is a function of \(x_{1}, \ldots, x_{6}\) fulfilling the functional equation:

\[
H(x'_{1}, \ldots, x'_{6}) = \varepsilon |J|^{r} H(x_{1}, \ldots, x_{6}),
\]

(6)

where

\[
\varepsilon = \begin{cases} 
1 & \text{for a Weyl density}, \\
\text{sgn} J & \text{for an ordinary density}.
\end{cases}
\]

(7)

Let us assume that \(H(x_{1}, \ldots, x_{6})\) is of class \(C^{1}\). For simplicity we introduce the following notation:

\[
H_{i} = \frac{\partial H}{\partial x_{i}}, \quad i = 1, 2, \ldots, 6.
\]

(8)

First we differentiate the functional equation (6) with respect to the parameters \(a_{ij}, i, j = 1, 2, 3\), and next we substitute

\[
[a_{ij}] = [\delta_{ij}],
\]

(9)

where \(\delta_{ij}\) is the Kronecker delta.

Then we get a system of nine equations of the first order with one unknown function \(H\) depending on six variables \(x_{i}, i = 1, \ldots, 6:\)

\[
\begin{align*}
2x_{1} H_{1} + 2x_{2} H_{2} + 2x_{3} H_{3} + x_{4} H_{4} + x_{5} H_{5} + x_{6} H_{6} &= -rH, \\
2x_{1} H_{1} + 2x_{2} H_{2} + 2x_{3} H_{3} + x_{4} H_{4} + x_{5} H_{5} + x_{6} H_{6} &= -rH, \\
2x_{2} H_{2} + 2x_{3} H_{3} + x_{4} H_{4} + x_{5} H_{5} + 2x_{6} H_{6} &= -rH, \\
2x_{3} H_{3} + x_{4} H_{4} + x_{5} H_{5} + 2x_{6} H_{6} &= 0, \\
2x_{5} H_{5} + x_{6} H_{6} &= 0,
\end{align*}
\]

(10)

\[
\begin{align*}
2x_{6} H_{6} &= 0, \\
2x_{6} H_{6} + x_{2} H_{2} + x_{3} H_{3} &= 0, \\
2x_{6} H_{6} + x_{4} H_{4} + x_{5} H_{5} &= 0, \\
2x_{6} H_{6} + x_{2} H_{2} + x_{3} H_{3} &= 0, \\
2x_{6} H_{6} + x_{4} H_{4} + x_{5} H_{5} &= 0, \\
2x_{6} H_{6} + x_{2} H_{2} + x_{3} H_{3} &= 0, \\
2x_{6} H_{6} + x_{4} H_{4} + x_{5} H_{5} &= 0, \\
2x_{6} H_{6} + x_{2} H_{2} + x_{3} H_{3} &= 0, \\
2x_{6} H_{6} + x_{4} H_{4} + x_{5} H_{5} &= 0.
\end{align*}
\]
We assert that (10) is a complete system. Denoting the left-hand sides of (10) by \( X_1, X_2, \ldots, X_9 \) respectively, we have

\[
X_i(H) = \sum_{k=1}^{6} a_{ik} H_k, \quad i = 1, 2, \ldots, 9,
\]

where the coefficients \( a_{ik} \) are certain simple functions of \( x_1, \ldots, x_4 \).

Let us introduce a shorter notation for Poisson brackets:

\[
(X_i, X_j) = \sum_{k=1}^{6} [X_i(a_{jk}) - X_j(a_{ik})] H_k, \quad i < j, i, j = 1, 2, \ldots, 9.
\]

After a number of simple operations based on (10), (11) and (12) we get

\[
(X_1, X_2) = 0, \quad (X_2, X_3) = 0, \quad (X_3, X_6) = X_8,
\]

\[
(X_1, X_3) = 0, \quad (X_2, X_4) = X_4, \quad (X_3, X_4) = 0,
\]

\[
(X_1, X_4) = -X_4, \quad (X_2, X_5) = 0, \quad (X_3, X_7) = X_7,
\]

\[
(X_1, X_5) = -X_5, \quad (X_2, X_6) = 0, \quad (X_3, X_8) = -X_8,
\]

\[
(X_1, X_6) = X_6, \quad (X_2, X_7) = X_7, \quad (X_3, X_9) = -X_9,
\]

\[
(X_1, X_7) = 0, \quad (X_2, X_8) = 0, \quad (X_4, X_8) = 0,
\]

\[
(X_1, X_8) = X_8, \quad (X_2, X_9) = X_9, \quad (X_4, X_9) = X_2 - X_1,
\]

\[
(X_1, X_9) = 0, \quad (X_3, X_4) = 0, \quad (X_4, X_7) = X_5,
\]

\[
(X_4, X_9) = -X_9, \quad (X_5, X_8) = X_3 - X_1, \quad (X_8, X_9) = X_8,
\]

\[
(X_4, X_9) = 0, \quad (X_5, X_9) = X_4, \quad (X_7, X_8) = X_6,
\]

\[
(X_5, X_9) = -X_7, \quad (X_6, X_7) = 0, \quad (X_7, X_9) = X_3 - X_2,
\]

\[
(X_5, X_7) = 0, \quad (X_6, X_8) = 0, \quad (X_8, X_9) = 0,
\]

from which it follows that (10) is a complete system.

For integrating complete systems of the type (10) it is convenient to find a so-called integrating direction ([15] or [10]).

Denoting the equations of the system (10) by (10.1)-(10.9), we shall integrate them in the following direction: (10.7), (10.5), (10.6), (10.8), (10.9), (10.4), (10.1), (10.2), (10.3).

To equation (10.7) corresponds the system of the ordinary equations:

\[
\frac{dx_1}{2x_4} = \frac{dx_2}{0} = \frac{dx_3}{0} = \frac{dx_4}{x_2} = \frac{dx_5}{x_6} = \frac{dx_6}{0}.
\]

Solving this system, we obtain

\[
H = \varphi(x_2, x_3, x_6, x_4^2 - x_1 x_2, x_4 x_6 - x_2 x_6) = \varphi(y_1, \ldots, y_6),
\]

where \( \varphi \in C^1 \).
Substituting solution (15) in equation (10.5) we get

\[ \frac{dy_1}{0} = \frac{dy_2}{0} = \frac{dy_3}{0} = \frac{dy_4}{2y_5} = \frac{dy_5}{y_5^2 - y_1y_2}. \]

Hence, by the assumption that \( y_5^2 - y_1y_2 \neq 0 \), we have

\[ H = \psi[x_2, x_3, x_6, x_2(x_1x_6^2 + x_2x_6^2 + x_3x_6^2 - x_1x_2x_3 - 2x_4x_5x_6)] \]
\[ = \psi(z_1, z_2, z_3, z_4), \]

where \( \psi \in C^1 \).

Substituting solution (17) in equation (10.6) we get the system of equations

\[ \frac{dz_1}{0} = \frac{dz_2}{2z_3} = \frac{dz_3}{z_1} = \frac{dz_4}{0}. \]

The solution of (18) is

\[ H = \theta[x_2, x_6^2 - x_2x_3, x_2(x_1x_6^2 + x_2x_6^2 + x_3x_6^2 - x_1x_2x_3 - 2x_4x_5x_6)] \]
\[ = \theta(u_1, u_2, u_3), \]

where \( \theta \in C^1 \).

Substituting solution (19) in equation (10.8) we get

\[ (x_4x_6 - x_2x_5) \theta_2 = 0, \]

hence, by assuming that \( x_4x_6 - x_2x_5 \neq 0 \), we receive

\[ \theta_2 = 0, \]

so

\[ H = \omega[x_2, x_2(x_1x_6^2 + x_2x_6^2 + x_3x_6^2 - x_1x_2x_3 - 2x_4x_5x_6)] \]
\[ = \omega(v_1, v_2), \quad \omega \in C^1. \]

Substituting solution (21) in equation (10.9) we get

\[ \frac{dv_1}{v_1} = \frac{dv_2}{v_2}. \]

The solution of (22) has the form

\[ H = \kappa(x_1x_2x_3 + 2x_4x_5x_6 - x_2x_5^2 - x_1x_2^2 - x_3x_4^2) = \kappa(w), \]

where \( \kappa(w) \in C^1 \).

Solution (23) can be rewritten in the form

\[ H = \kappa(w) = \kappa \begin{pmatrix} x_1 & x_4 & x_6 \\ x_4 & x_2 & x_6 \\ x_6 & x_4 & x_3 \end{pmatrix}. \]

Substituting solution (23) in equation (10.4) we obtain an identity,
and thus equation (10.4) is not independent of the previously integrated equations.

Substituting solution (24) in (10.1) we have

\( 4x'(w)w = -r x(w). \)  

Solving the homogeneous equation (25) we obtain (similarly to the equation (21), Section 2)

\( H = C |w|^{-r/4}, \quad r - \text{arbitrary}, \ w \neq 0, \)

where the integration constant \( C \neq 0 \) has the form

\[ C = \begin{cases} 
  C_1 & \text{for } w > 0, \\
  C_2 & \text{for } w < 0, 
\end{cases} \]

and

\( w = \begin{vmatrix} 
  x_1 & x_4 & x_5 \\
  x_4 & x_2 & x_6 \\
  x_5 & x_6 & x_3 
\end{vmatrix}. \)

We verify without difficulty that solution (25) fulfils equations (10.2) and (10.3).

We also verify that the symmetric determinant (28), formed from the essential components (1) of the tensor \( R_{a\beta\gamma\delta} \), is a Weyl density of weight (4), i.e. denoting by \( w' \) the right-hand side of (28) for \( x_\nu, i = 1, 2, \ldots, 6 \) of the form (5) we obtain (after tedious calculations)

\[ w' = J^{-4}w. \]

The results of Section 3 can now be formulated as follows:

**Theorem 2.** In the space \( V_3 \) each scalar concomitant \( H(x_1, \ldots, x_6) \) (of class \( C^1 \)) of the curvature tensor \( R_{a\beta\gamma\delta} \) is a constant function \( H(x_1, \ldots, x_6) = C \), while every density concomitant of weight \(-r\) is a Weyl density of the form

\[ H(x_1, \ldots, x_6) = C |w|^{-r/4}, \quad C \neq 0, \ w \neq 0, \]

where \( w \) is defined by (28) and \( C \) by (27).

**Remark 1.** The above considerations have only been based on the symmetry and antisymmetry of tensor \( R_{a\beta\gamma\delta} \) but we have ignored the fact that \( R_{a\beta\gamma\delta} \) as a curvature tensor comes from the metric tensor \( g_{a\beta} \). The whole consideration is maintained if we assume that the tensor \( R_{a\beta\gamma\delta} \) has only properties (10) and is independent of the tensor \( g_{a\beta} \), i.e. the assumption that the space is Riemannian is not necessary.

**Remark 2.** The concomitant defined by formula (28) being an algebraic concomitant of the tensor \( R_{a\beta\gamma\delta} \), it can be called a differential
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concomitant of the second order of the metric tensor \( g_{\alpha \beta} \) (because \( R_{\alpha \beta \gamma} \) are expressed by \( g_{\alpha \beta}, \partial_\gamma g_{\alpha \beta}, \partial_\beta g_{\alpha \gamma} \)). However, there are other algebraic concomitants of the tensor \( g \), which are densities. The simplest of those is \( \det(g_{\alpha \beta}) \), another one (for \( n = 3 \)) is a symmetric determinant of the third order formed from the essential components of the tensor

\[
G_{\alpha \beta \gamma} = 2 \frac{\partial}{\partial t} g_{[\alpha \gamma]} g_{\beta \beta'},
\]

i.e. from the minors of the second order of \( \det(g_{\alpha \beta}) \). The tensor \( G_{\alpha \beta \gamma} \) of the form (31) is the so-called induced metric tensor of the bivector space \( V_2^n \), occurring in paper [8].

References

[3] L. Bieszk, O komitantach algebraicznych tensora krzywejnego \( R^\alpha_{\beta \gamma} \), w przestrzeniach dwuwymiarowych, i trójwymiarowych, wyjściowych z koniecznością afiniczną (unpublished).
[5] — — O komitantach algebraicznych liniowych tensora \( T^\alpha_{\beta \gamma} \) w przestrzeni dwuwymiarowej, ibidem 80 (1967), p. 105-111.

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