OPERATOR SEMI-STABLE PROBABILITY MEASURES
ON BANACH SPACES

BY

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In this paper we define operator semi-stable probability measures on a real separable Banach space which are identified as limit laws. Further, we get a representation of the characteristic functionals of operator semi-stable probability measures.

1. Notation and preliminaries. Let $X$ denote a real separable Banach space with norm $\| \cdot \|$ and with dual space $X^*$. By $\langle \cdot, \cdot \rangle$ we denote the dual pairing between $X$ and $X^*$. Further, $B(X)$ will denote the algebra of continuous linear operators on $X$ with norm topology. Given a subset $F$ of $B(X)$, we denote by $\text{Sem}(F)$ the closed multiplicative semigroup of operators spanned by $F$. The unit and zero operators will be denoted by $I$ and $0$, respectively.

A sequence $\{\mu_n\}$ of probability measures on $X$ is said to converge to a probability measure $\mu$ on $X$ if for every bounded continuous real-valued function $f$ on $X$

$$\int_X fd\mu_n \rightarrow \int_X fd\mu.$$ 

The characteristic functional of $\mu$ is defined on $X^*$ by

$$\mu(y) = \int_X e^{i\langle x,y \rangle} \mu(dx),$$

where $y \in X^*$. For an operator $A$ from $B(X)$ and a probability measure $\mu$ on $X$ let $A$ denote the probability measure defined by $A\mu(E) = \mu(A^{-1}(E))$ for all Borel subsets $E$ of $X$. It is easy to check the equations

$$A(\mu \ast \nu) = A\mu \ast A\nu, \quad A\mu(y) = \hat{\mu}(A^*y),$$

where $A^*$ denotes the adjoint operator. Moreover, $A_n\mu_n \rightarrow A\mu$ whenever $A_n \rightarrow A$ and $\mu_n \rightarrow \mu$. A probability measure $\mu$ on $X$ is said to be full if
its support is not contained in any proper hyperplane of $X$. By $\delta_x (x \in X)$
we denote the probability measure concentrated at the point $x$.

A probability measure $\mu$ on $X$ is said to be infinitely divisible whenever
for every positive integer $n$ there exists a probability measure $\mu_n$ such that
$\mu = \mu_n^n$, where the power is taken in the sense of convolution. Let $\mu$
be an infinitely divisible probability measure on $X$. Then for every $\sigma \geq 0$
there exists an infinitely divisible measure $\nu$ on $X$ such that $\hat{\nu}(y) = [\mu(y)]^\sigma$.
We denote $\nu$ by $\mu^\sigma$. The set $\{\mu^\sigma\}_{\sigma \geq 0}$ is an Abelian semigroup with the con-
volution as a semigroup operation, and the mapping $\sigma \rightarrow \mu^\sigma$ is a continuous
homomorphism of the additive semigroup of non-negative real numbers
onto $\{\mu^\sigma\}_{\sigma \geq 0}$ (Proposition 1.2 of [8]).

**Lemma 1.** Let $\mu$ and $\nu$ be probability measures on $X$ and let $\{x_n\}$ be
a sequence of elements of $X$ such that

$$
\lim_{n \to \infty} \exp(i \langle x_n, y \rangle) = \hat{\nu}(y) \quad \text{for all } y \in X^*.
$$

Then there exists a unique element $x \in X$ such that $\mu = \delta_x * \nu$.

The lemma follows immediately from Lemma 1.1 of [8].

Given a probability measure $\mu$ on $X$, we define $\bar{\mu}$ by $\bar{\mu}(E) = \mu(-E)$,
where $-E = \{-x: x \in E\}$. For any probability measure $\mu$ on $X$ the measure $|\mu|^2 = \mu * \bar{\mu}$ is called the symmetrization of $\mu$.

Let $[a]$ be the largest integer not greater than $a$.

2. **Stating the problem.** Let $\mu$ be a probability measure on $X$. We call $\mu$
operator semi-stable if its characteristic functional $\hat{\mu}$ satisfies the functional
equation

$$(2.1) \quad [\hat{\mu}(y)]^\sigma = \hat{\mu}(B^*y)e^{\langle b, y \rangle} \quad \text{for all } y \in X^*,
$$

where $B \in B(X)$, $b \in X$ and $\sigma \in (0, 1)$. In the one-dimensional case, characteristic functionals which satisfy for all $x$ an equation of the form

$$
\varphi(x) = [\varphi(bx)]^\sigma,
$$

where $a > 0$ and $0 < b < 1$, have been considered by Lévy ([11], p. 204)
and the solutions have been called by him semi-stable. Semi-stable measures
on the real line have been studied by Kruglov in [9], by Pillai in [13]
and by Rao and Ramachandran in [14]. Operator semi-stable measures
on finite-dimensional spaces have been considered by Jajte in [6]. Kumar
[10] has treated semi-stable measures on Hilbert spaces and proved that
they are limit laws. We obtained a representation of the characteristic
functionals of these laws in the same manner as Jajte did in [5] for stable
probability measures.

**Proposition 1.** Every operator semi-stable measure on $X$ is infinitely
divisible.
Proof. Let $N$ be a collection of closed subspaces of $X$ with finite codimension and let $p_N: X \to X/N$, $N \in N$, be canonical maps.

Let $\mu$ be an operator semi-stable measure on $X$ such that

$$\hat{\mu}(y) = \mu(B^*y)e^{i\theta(y)}$$

for all $y \in X^*$,

where $B \in B(X)$, $b \in X$ and $c \in (0, 1)$.

Let $\nu = |\mu|^2$ and $N \in N$. We have

$$\hat{\nu}(y) = \left[\hat{\nu}(B^*y)^{1/e_k}\right]^{1/e_k}$$

(y \in X^*)

and

$$\hat{\nu}(\pi_N y) = \left[\hat{\nu}(B^*y^{\pi_N y})^{1/e_k}\right]^{1/e_k}$$

(y \in (X/N)^*)

for $k = 1, 2, \ldots$ Let $n_k = [e^{-k}]$ and $y \in (X/N)^*$. If $\hat{\nu}(\pi_N y) = 0$, then $\hat{\nu}(B^*y^{\pi_N y}) = 0$ for $k = 1, 2, \ldots$

Let $\hat{\nu}(\pi_N y) \neq 0$. Since

$$\left|\hat{\nu}(\pi_N y) - \hat{\nu}(B^*y^{\pi_N y})^{1/e_k}\right|^{1/e_k} \leq \left|\hat{\nu}(B^*y^{\pi_N y})^{1/e_k} - 1\right|$$

$$= 1 - \nu(B^*y^{\pi_N y})^{e_k(e^{-k} - e^{-k})} = 1 - \nu(B^*y^{\pi_N y})^{1 - e_k(e^{-k} - e^{-k})},$$

$\hat{\nu}(B^*y^{\pi_N y})$ converges to $\nu(\pi_N y)$. Thus $\pi_N B^k(y)^{\pi_N y} \to \nu$. Hence $\pi_N \nu$ is an infinitely divisible measure and $\hat{\mu}(y) \neq 0$ for all $y \in X^*$.

We have

$$\mu(y) = \left[\mu(B^*y)^{1/e_k}\right]^{1/e_k}\exp\left(i \frac{1}{e_k} \langle a_k, y \rangle\right)$$

for all $y \in X^*$,

where

$$a_k = e^k \sum_{j=1}^{k} \frac{1}{e^j} B^j b \quad (B^0 = I), k = 1, 2, \ldots$$

If $N \in N$, then

$$\hat{\mu}(\pi_N y) = \left[\hat{\mu}(B^*y^{\pi_N y})^{1/e_k}\right]^{1/e_k}\exp\left(i \frac{1}{e_k} \langle \pi_N a_k, y \rangle\right)$$

for $y \in (X/N)^*$.

Since

$$|\hat{\mu}(\pi_N y) - \hat{\mu}(B^*y^{\pi_N y})|^{1/e_k} \exp\left(i \langle n_k \pi_N a_k, y \rangle\right)$$

$$\leq \left|\left|\hat{\mu}(B^*y^{\pi_N y})\exp\left(i \langle \pi_N a_k, y \rangle\right)\right|^{e_k^{-k} - e_k^{-k}} - 1\right|$$

$$= \left|\hat{\mu}(\pi_N y)^{e_k} - 1\right| = \left|\hat{\mu}(\pi_N y)^{1 - e_k(e^{-k} - e^{-k})} - 1\right|,$$

$$\left|\hat{\mu}(B^*y^{\pi_N y})\exp\left(i \langle n_k \pi_N a_k, y \rangle\right)\right|$$

converges to $\hat{\mu}(\pi_N y)$ for every $y \in (X/N)^*$. Thus

$$\pi_N B^k \mu^{\pi_N y} \to \pi_N \mu$$

and $\pi_N \mu$ is infinitely divisible for all $N \in N$. By Theorem 1.1.9 of [3], $\mu$ is infinitely divisible. This completes the proof of the proposition.
3. Characterization of operator semi-stable measures. The following theorem proves that operator semi-stable probability measures on $X$ are limit laws.

**Theorem 1.** A probability measure $\mu$ on $X$ is an operator semi-stable measure if and only if there exist a probability measure $\nu$ on $X$, an operator $B \in B(X)$, sequences $\{a_k\}$ and $\{n_k\}$ of elements of $X$ and of positive integers, respectively, such that, for certain $c \in (0, 1)$,

$$(3.1) \quad \lim_{k \to \infty} \frac{n_k}{n_{k+1}} = c$$

and

$$(3.2) \quad B^k \mu^{n_k} \delta_{a_k} \to \mu.$$  

**Proof. Necessity.** Suppose that $\mu$ is an operator semi-stable measure and

$$[\hat{\mu}(y)]^e = \hat{\mu}(B^* y) e^{i(b, y)} \quad \text{for all } y \in X^e,$$

where $B \in B(X)$, $c \in (0, 1)$ and $b \in X$.

Let

$$n_k = [c^{-k}] \quad \text{and} \quad a_k = c^k \sum_{j=1}^{k} \frac{1}{c^j} B^j B^{-1} b \quad (B^0 = I).$$

We have

$$\lim_{n \to \infty} \frac{n_k}{n_{k+1}} = c.$$  

Since

$$(3.3) \quad \hat{\mu}(y) = [\hat{\mu}((B^*)^k y)]^{1/c^k} \exp \left( \frac{1}{c^k} \langle a_k, y \rangle \right)$$

$$= [\hat{\mu}((B^*)^k y)]^{1/c^k} \exp \left( i \langle n_k a_k, y \rangle \right) [\hat{\mu}((B^*)^k y)]^{1/c^k - n_k},$$

the sequence $\{B_k \mu^{n_k} \delta_{n_k a_k}\}$ is shift compact (Theorem 3.2.2 of [12]).

By Lemma 1.2.4 of [3] we show that

$$(3.4) \quad \lim_{k \to \infty} \sup_{y \in U_r} |B_k \mu^{n_k} \delta_{n_k a_k}(y) - \mu(y)| = 0 \quad \text{for all } r > 0,$$

where $U_r = \{x \in X : \|x\| \leq r\}$ and $U_r^c = \{y \in X : \langle x, y \rangle \leq 1 \text{ for all } x \in U_r\}$.

We have

$$|\hat{\mu}(y) - B^k \mu^{n_k} \delta_{n_k a_k}(y)| \leq |\hat{\mu}((B^*)^k y) \exp(i \langle a_k, y \rangle)|^{c^k - [c^{-k}] - 1}$$

$$= |[\mu^{c^k}(y)]^{c^k - [c^{-k}] - 1} - 1| = |[\hat{\mu}(y)]^{c^k - [c^{-k}] - 1} - 1|.$$

By $\mu^{1 - c^k [c^{-k}]} \to \delta_0$ (Proposition 1.2 of [8]) and by Lemma 1.2.3 of [3], condition (3.4) holds. Thus the sequence $\{B_k \mu^{n_k} \delta_{n_k a_k}\}$ converges to $\mu$.  

Sufficiency. Assume that there exist a probability measure \( \nu \) on \( X \), an operator \( B \in B(X) \), sequences \( \{a_k\} \) and \( \{n_k\} \) of elements of \( X \) and of positive integers, respectively, such that (3.1) and (3.2) hold. Further, the sequence

\[
\left\{ \hat{\nu} (B^* (B^*)^k y) \right\} \exp \left( \left( \frac{n_k}{n_{k+1}} a_{k+1} - B a_k, y \right) \right) \exp \left( i \left( \frac{n_k}{n_{k+1}} a_{k+1} - B a_k, y \right) \right)
\]

converges to \( [\hat{\mu} (y)]^c \) for all \( y \in X^* \). By (3.2) we have

\[
B(B^* \nu^{n_k} \delta_{a_k}) \to B \mu.
\]

Clearly,

\[
\hat{\mu}(B^* y) \exp \left( i \left( \frac{n_k}{n_{k+1}} a_{k+1} - B a_k, y \right) \right) \to [\hat{\mu} (y)]^c \quad \text{for all } y \in X^*.
\]

By Lemma 1 there exists a \( b \in X \) such that \( \mu^c = B \mu^c \delta_b \), which completes the proof of the theorem.

Given a probability measure \( \mu \) on \( X \), we denote by \( C_p(\mu) \) \((0 < p < \infty)\) the subset of \( B(X) \) consisting of all invertible operators \( A \) with the property \( [\hat{\mu} (y)]^p = A \mu^c \delta_a (y) \) for all \( y \in X^* \) and certain \( a \in X \). Let

\[
C(\mu) = \{ p \in (0, \infty) : C_p(\mu) \neq \emptyset \}.
\]

It is clear that if \( C(\mu) \neq \{1\} \), then \( \mu \) is an operator semi-stable measure.

**Proposition 2.** Let \( \mu \) be a probability measure with \( C(\mu) \neq \{1\} \). Then either \( C(\mu) = \{ s^n : n \in \mathbb{Z} \} \) for certain \( s \in (0, 1) \) or the set \( C(\mu) \) is dense in \((0, \infty)\).

**Proof.** We assume that \( \sup C(\mu) \cap (0, 1) = s < 1 \). Suppose that \( C_s = \emptyset \). Then there exist \( p, q \in C(\mu) \cap (0, 1) \) such that \( s^2 < p < q < s \).

Further, we get \( s < p/q < 1 \) and \( C^{p/q-1} \neq \emptyset \), which contradicts the assumption that \( s \) is the supremum of \( C(\mu) \cap (0, 1) \). Suppose now that \( C(\mu) \neq \{ s^n : n \in \mathbb{Z} \} \). Then there exists an \( r \in C(\mu) \cap (0, s) \) such that \( r \neq s^n \)

for \( n = 1, 2, \ldots \). For some positive integer \( n_0 \) we have \( s^{n_0+1} < r < s^{n_0} \).

Hence \( s < r/s^{n_0} < 1 \) and \( C_{r/s^{n_0}} \neq \emptyset \), which contradicts the assumption that \( s \) is the supremum of \( C(\mu) \cap (0, 1) \). The proposition is proved.

**Theorem 2.** Let \( \mu \) be a full probability measure on \( X \). Then there exists an operator \( B \in B(X) \) with

\[
\lim_{t \to 0} \exp (B \log t) = 0
\]

such that

\[
\mu^t = \exp (B \log t) \mu \delta b_0
\]

for all \( t > 0 \),

where \( b_0 \in X \), if and only if there exist sequences \( \{ B_n \} \) and \( \{ c_n \} \) of operators of the algebra \( B(X) \) and of real numbers of \((0, 1)\), respectively, such that
Sem($\{B_n : n = 1, 2, \ldots\}$) is compact in the norm topology of $B(X)$, $c_n \to 1$ and

$$\mu^n = B_n \mu \ast \delta_{b_n} \quad \text{for } n = 1, 2, \ldots \text{ and } b_n \in X.$$  

The theorem follows immediately from Theorem 3.1 of [8].

4. Representation of operator semi-stable measures. For the theory of infinitely divisible probability measures on Banach spaces and even on more general algebraic structures we refer to [15] and [3]. In particular, if $F$ is any bounded non-negative Borel measure, then $e(F)$ is defined as

$$e(F) = e^{-F(X)} \sum_{k=0}^{\infty} \frac{1}{k!} F^{*k}, \quad \text{where } F^{*0} = \delta_0.$$  

The measure $F$ is called a Poisson exponent of $e(F)$. Let $M$ be a not necessarily bounded Borel measure on $X$ vanishing at 0. If there exists a representation $M = \sup F_n$, where $F_n$ are bounded and the sequence \{e($F_n$)\} of associated Poisson measures is shift compact, then each cluster point of the sequence \{e($F_n$) \ast \delta_{x_n}\} ($x_n \in X$) is called a generalized Poisson measure and is denoted by $\tilde{\delta}(M)$. Clearly, $\tilde{\delta}(M)$ is uniquely determined up to translation, i.e. for two cluster points, say $\mu_1$ and $\mu_2$, of \{e($F_n$) \ast \delta_{x_n}\} and \{e($F_n$) \ast \delta_{y_n}\}, respectively, we have $\mu_1 = \mu_2 \ast \delta_x$ for certain $x \in X$ ([15], p. 313). Further, the measure $M$ is called a generalized Poisson exponent of $\tilde{\delta}(M)$. Let $M(X)$ denote the set of all generalized Poisson exponents of $X$.

By a Gaussian measure on $X$ we mean a probability measure $\varrho$ on $X$ such that for every $y \in X^*$ the measure $\varrho_y$ induced on the real line is Gaussian. In this paper we consider only symmetric Gaussian measures. For such measures the characteristic functional is of the form

$$\hat{\varrho}(y) = \exp \left(-\frac{1}{2} \langle y, Ry \rangle \right) \quad (y \in X^*),$$  

where $R$ is the covariance operator, i.e. a compact operator from $X^*$ into $X$ with the properties $\langle y_1, Ry_2 \rangle = \langle y_2, Ry_1 \rangle$ for all $y_1, y_2 \in X^*$ (symmetry) and $\langle y, Ry \rangle \geq 0$ (non-negativity) (see [17], p. 136, and [2]). By $R(X)$ we denote the set of all covariance operators of Gaussian measures on $X$.

Tortrat proved in [15] (see also [3]), the following analogue of the Lévy-Khinchine representation of infinitely divisible laws: each infinitely divisible measure $\mu$ on $X$ has a unique representation $\mu = o \ast \tilde{\delta}(M)$, where $o$ is a symmetric Gaussian measure on $X$ and $M \in M(X)$.

**Proposition 3.** Let $B \in B(X)$. Then a probability measure $\mu$ on $X$ is operator semi-stable with $\mu^c = B\mu \ast \delta_b$ for some $c \in (0, 1)$ and $b \in X$ if
and only if \( \mu = \varphi \ast \delta(M) \), where \( \varphi \) is a symmetric Gaussian measure with the covariance operator \( R \) and \( M \in M(X) \) such that \( cM = BM \) and \( cR = BRB^* \).

The proof is trivial.

**Corollary 1.** Let \( B \in B(X) \) and let \( \mu \) be an operator semi-stable probability measure on \( X \) with \( \mu^c = B\mu \ast \delta_b \) for some \( c \in (0, 1) \) and \( b \in X \). If \( \mu = \varphi \ast \delta(M) \), where \( \varphi \) is a symmetric Gaussian measure and \( M = M(X) \), then \( \varphi \) and \( M \) are concentrated on subspaces \( X_1 \) and \( X_2 \), respectively, which are invariant under \( B \).

Let \( B \) be an invertible operator from \( B(X) \) with

\[
\lim_{n \to +\infty} B^n = 0.
\]

Given a subset \( E \) of \( X \), we put \( \tau(B) = \{ B^n x : x \in E, n \in \mathbb{Z} \} \). It is clear that for any compact set with the property \( 0 \notin E \) and for any pair \( r_1, r_2 \) \((r_1 < r_2)\) of positive numbers the inequality \( r_1 \leq \|B^n x_k\| \leq r_2 \) \((x_k \in X)\) implies the boundedness of the sequence \( \{n_k\} \). This simple fact yields the following

**Lemma 2.** Let \( E \) be a compact subset of \( X \) and \( 0 \notin E \). Then for every pair \( r_1, r_2 \) \((r_1 \leq r_2)\) of positive numbers the set \( \{ x : r_1 \leq \|x\| \leq r_2 \} \cap \tau(B) \) is compact.

The following lemma reduces our problem of examining a measure \( M \in M(X) \) with the property \( cM = BM \) for some \( c > 0 \) to the case of measures concentrated on \( \tau(E) \), where \( E \) is compact and \( 0 \notin E \).

**Lemma 3.** Let \( M \in M(X) \) and \( cM = BM \) for certain \( c > 0 \). Then there exists a decomposition

\[
M = \sum_{n=1}^{\infty} M_n,
\]

where \( M_n \in M(X) \), \( cM_n = BM_n \), \( M_n \) are concentrated on disjoint sets \( \tau(E_n) \), \( 0 \notin E_n \) and \( E_n \) are compact.

The lemma follows immediately from Lemma 5.4 of [16].

Now, we are ready to prove the representation of the characteristic functionals of operator semi-stable measures.

**Theorem 3.** Let \( B \) be an invertible operator from \( B(X) \) with

\[
\lim_{n \to +\infty} B^n = 0.
\]

A probability measure \( \mu \) on \( X \) is an operator semi-stable measure and \( \mu^c = B\mu \ast \delta_b \), where \( c \in (0, 1) \) and \( b \in X \), if and only if there exist an element \( a \in X \), an operator \( R \in \mathcal{R}(X) \) such that \( cR = BRB^* \) for certain \( c \in (0, 1) \).
and a finite measure \( \lambda \) on \( T = \{ x \in X : 1 \leq \| x \| \leq \| B^{-1} \| \} \) such that

\[
\hat{\mu}(y) = \exp \left\{ i \langle a, y \rangle - \frac{1}{2} \langle y, By \rangle + \sum_{n=-\infty}^{\infty} \frac{1}{\sigma^n} \int_T \left[ \exp(i \langle B^n x, y \rangle) - 1 - i \langle B^n x, y \rangle 1_D(B^n x) \right] \lambda(dx) \right\},
\]

where \( 1_D \) denotes the indicator of the unit ball \( D \) in \( X \) and \( y \in X^* \).

Proof. To prove the necessity let us assume that \( \mu \) is an operator semi-stable measure, \( B \) is an invertible operator from \( B(X) \) with

\[
\lim_{n \to \infty} B^n = 0
\]

and \( \mu^c = B \mu * \delta_0 \) for certain \( c \in (0, 1) \). Further, \( \mu \) is an infinitely divisible measure and \( \mu = q * \tilde{\sigma}(M) \), where \( q \) is a symmetric Gaussian measure with the covariance operator \( E \) and \( M \in M(X) \). Moreover, for certain \( c \in (0, 1) \) we have

\[
BM = cM, \quad cR = BBR^*.
\]

By Lemma 3 there exists a decomposition

\[
M = \sum_{n=1}^{\infty} M_n,
\]

where \( M_n \in M(X) \), \( BM = cM \), \( M_n \) are concentrated on disjoint sets \( \tau(E_n), 0 \notin E_n \) and \( E_n \) are compact.

Let \( D_n = \tau(E_n) \cap \{ x : 1 \leq \| x \| \leq \| B^{-1} \| \} \). By Lemma 2 the set \( D_n \) is compact. We define an equivalence relation in \( D_n \) as follows: \( x_1 \sim x_2 \), \( x_1, x_2 \in D_n \), if and only if there exists an integer \( n \) such that \( x_1 = B^n x_2 \). In order to prove the continuity of this relation suppose that \( x_n \sim x_n' \) and that the sequences \( \{ x_n \} \) and \( \{ x_n' \} \) converge to \( x \) and \( x' \), respectively. Then for some integers \( k_n \) we have \( B^{k_n} x_n = x_n' \). By the compactness of \( E_n \) and the assumption \( 0 \notin E_n \), the sequence \( \{ k_n \} \) is bounded. Clearly, for any its cluster point \( k_0 \) we have \( B^{k_0} x = x' \), which implies \( x \sim x' \). Thus the relation \( \sim \) is continuous. Hence it follows that the quotient space \( D_n / \sim \) is compact ([1], p. 97). The coset containing \( x \) will be denoted by \( [x] \).

Further, the mapping \( x \rightarrow [x] \) from \( D_n \) onto \( D_n / \sim \) is continuous. A theorem of Kuratowski (Theorem 1.4.2 of [12]) shows that there exists a Borel subset \( T_n \) of \( D_n \) such that \( T_n \) intersects each \( [x] \) at exactly one point.

Let \( f_n \) be a mapping of \( \bigcup_{n=1}^{\infty} T_n \times Z \) into \( \tau(E_n) \) such that \( f_n(x, n) = B^n x \). The mapping \( f_n \) is continuous and one-one. By a theorem of Kuratowski (Corollary 1.3.2 of [12]) the mapping \( f_n^{-1} \) is measurable. Let \( f \) be a mapping of \( \bigcup_{n=1}^{\infty} T_n \times Z \) into \( \bigcup_{n=1}^{\infty} \tau(E_n) \) such that \( f(x, m) = f_n(x, m) \) if \( x \in T_n \). The
mapping \( f \) is one-one, and \( f \) and \( f^{-1} \) are measurable. Hence the \( \sigma \)-field generated by the collection of the sets \( B^n(F) \), where \( n \) is integer and \( F \) stands for Borel subsets of \( T_\circ = \bigcup_{n=1}^{\infty} T_n \), consists of all Borel subsets of \( \bigcup_{n=1}^{\infty} x(E_n) \).

Put

\[
g(n, F) = M(\{B^n x : x \in F\}) \quad (n \in \mathbb{Z}).
\]

Since \( BM = cM \), we have

\[
g(n, F) = c^{-n}g(0, F) = c^{-n}\lambda_0(F),
\]

where \( \lambda_0(F) = g(0, F) \) for all Borel subsets of \( T_\circ \). We can extend (4.4) for all Borel subsets of \( X \setminus \{0\} \) by the formula

\[
M(F) = \sum_{n=-\infty}^{\infty} \frac{1}{c^n} \int_{T} 1_F(B^n x) \lambda(\mathcal{d}x),
\]

where \( \lambda(G) = \lambda_0(G \cap T_\circ) \) for any Borel subset \( G \) of \( T = \{x : 1 \leq \|x\| \leq \|B^{-1}\|\} \). Further, from the Dettweiler representation of the characteristic functionals of an infinitely divisible measure on \( X \) (Theorem 1.2.5 of [3]) we get the formula

\[
\mu(y) = \exp \left\{ i \langle a, y \rangle - \frac{1}{2} \langle y, By \rangle + \int_X \left[ e^{i \langle x, y \rangle} - 1 - i \langle x, y \rangle 1_D(x) \right] M(dx) \right\},
\]

where \( y \in X^* \), \( a \in X \), \( R \in R(X) \), \( M \in M(X) \) and \( 1_D \) denotes the indicator of the unit ball \( D \) in \( X \). Inserting (4.5) for \( M \) into (4.6) we get (4.1).

By a simple calculation we can check that each measure \( \mu \) with the characteristic functional of form (4.1) fulfills equation (2.1), which completes the proof.

A probability measure \( \mu \) on \( X \) is called semi-stable if its characteristic functional satisfies the functional equation

\[
[\hat{\mu}(y)]^p = \hat{\mu}(by)e^{i\langle a, y \rangle} \quad \text{for all } y \in X^*,
\]

where \( 0 < |b| < 1 \), \( 0 < c < 1 \) and \( a \in X \).

**Proposition 4.** Let \( \mu \) be a non-degenerate measure on \( X \) satisfying (4.7) and let \( p \) be the unique real solution of the equation \( |b|^p = c \). Then

(a) \( 0 < p < 2 \);

(b) \( p = 2 \) if and only if \( \mu \) is a Gaussian measure;

(c) \( 0 < p < 2 \) if and only if \( \mu = \delta(M) \) for some \( M \in M(X) \).
The proposition is an immediate consequence of the following fact: if \( \mu \) is a semi-stable measure of \( X \), then \( y \mu \) is a semi-stable measure on the real line for all \( y \in X^* \).

From now on the unique real solution \( p \) of the equation \(|b|^p = c\) for a non-degenerate semi-stable probability measure \( \mu \) on \( X \) will be called the exponent of \( \mu \).

**Corollary 2.** Let \( \mu \) be a probability measure on \( X \). Then \( \mu \) is semi-stable if and only if either \( \mu \) is Gaussian or there exist constants \( p \) (\( 0 < p < 2 \)) and \( b \) (\( 0 < |b| < 1 \)), a finite measure \( \lambda \) on \( T = \{ x : 1 \leq \|x\| \leq 1/|b| \} \) and an element \( a \in X \) such that, for every \( y \in X^* \),

\[
(4.8) \quad \hat{\mu}(y) = \exp \left\{ i \langle a, y \rangle + \sum_{n=-\infty}^{\infty} \frac{1}{|b|^{pn}} \int_T \left[ \exp \left( i b^n \langle x, y \rangle \right) - 1 - i b^n \langle x, y \rangle 1^D(b^n x) \right] \lambda(dx) \right\},
\]

where \( 1_D \) denotes the indicator of the unit ball \( D \) in \( X \).

The measure \( \lambda \) appearing in representation (4.8) will be called the representing measure for \( \mu \). Let \( \Lambda_p(X) \) denote the set of all representing measures corresponding to semi-stable measures on \( X \) with the exponent \( p \) (\( 0 < p < 2 \)). Clearly, \( \lambda \in \Lambda_p(X) \) if and only if the measure \( M \) defined by

\[
(4.9) \quad M(F) = \sum_{n=-\infty}^{\infty} \frac{1}{|b|^{pn}} \int_T 1_F(b^n x) \lambda(dx)
\]

belongs to \( M(X) \). The set \( M(X) \) has the following property: if \( N \) is a non-negative measure on \( X \) and \( N \leq M \), where \( M \in M(X) \), then \( N \in M(X) \). Hence \( \lambda \in \Lambda_p(X) \) if and only if the measure \( \lambda_0 \) defined by \( \lambda_0(E) = \lambda(E) + + \lambda(-E) \) belongs to \( \Lambda_p(X) \). This fact reduces the problem of determining \( \Lambda_p(X) \) to examining symmetric measures \( \lambda \). We say that \( X \) is of type \( r \) (\( 2 \geq r > 0 \)) whenever there exists a positive constant \( c \) such that for any collection \( \xi_1, \xi_2, \ldots, \xi_n \) of independent symmetrically distributed \( X \)-valued random variables we have

\[
E \left\| \sum_{j=1}^{n} \xi_j \right\|^r \leq c \sum_{j=1}^{n} E \left\| \xi_j \right\|^r.
\]

**Theorem 4.** If \( X \) is of type \( r \) and \( r > p \), then \( \Lambda_p(X) \) consists of all finite Borel measures on \( T \).

**Proof.** We use arguments similar to those given by Jurek and Urbanik in [7]. To prove the theorem it suffices to show that for each symmetric finite measure \( \lambda \) on \( X \) the measure \( M \) defined by (4.9) belongs
to $M(X)$. Let

$$M_0(F) = \sum_{n=-\infty}^{0} \frac{1}{|b|^{pn}} \int_{\mathcal{F}} 1_{F}(b^{n}x) \lambda(dx)$$

and

$$M_k(F) = \frac{1}{|b|^{pk}} \int_{\mathcal{F}} 1_{F}(b^{k}x) \lambda(dx) \quad (k = 1, 2, \ldots);$$

then the measures $M_n (n = 0, 1, 2, \ldots)$ are finite on $X$ and vanish at 0. Put, for simplicity, $\mu_k = e(M_k) \ (k = 0, 1, \ldots)$. Since

$$M = \sum_{k=0}^{\infty} M_k,$$

we conclude that $M \in M(X)$ if and only if the sequence $\{\mu_0 \ast \mu_1 \ast \ldots \ast \mu_n\}$ converges to a probability measure on $X$ or, equivalently, the series $\sum_{k=0}^{\infty} \eta_k$ of independent $X$-valued random variables $\eta_0, \eta_1, \ldots$ with probability distributions $\mu_0, \mu_1, \ldots$, respectively, converges almost surely (Theorem 3.1 of [4]). To prove that $\sum_{k=0}^{\infty} \eta_k$ converges almost surely, it suffices, by the Borel-Cantelli lemma, to show the convergence of the series

$$\sum_{k=0}^{\infty} \mu_k(\{x: \|x\| > a^k\}),$$

where $a = |b|^{(r+1)^{-1}(r-p)} < 1$. Setting $a_k = M_k(X)$ and $\nu_k = a_k^{-1} M$ for $k = 1, 2, \ldots$, we get

$$\mu_k = \exp(-a_k) \sum_{n=0}^{\infty} \frac{a_n^k}{n!} \nu_k^n$$

and

$$a_k = |b|^{-pk} \lambda(T).$$

Further, for a positive constant $\sigma$ we obtain

$$\int_{X} \|x\|^r \nu_k^n(dx) \leq \sigma_1 |b|^{krn}.$$

Consequently, by (4.11) we have

$$\int_{X} \|x\|^r \mu_k(dx) \leq \sigma_1 |b|^{kr} \exp(-a_k) \sum_{n=0}^{\infty} \frac{a_n^k}{(n-1)!}. $$
Since
\[ \exp(-a_k) \sum_{n=0}^{\infty} \frac{a_k^n}{(n-1)!} \leq c_n a_k \quad \text{for certain } c_n > 0, \]
we get the inequality
\[ \int_X \|x\|^r \mu_k(dx) \leq c_n a_k^{k+r+1} \quad (k = 1, 2, \ldots) \]
with a constant \( c_n \). Consequently,
\[ \mu_k(\{x: \|x\| > a_k\}) \leq a^{-kr} \int_X \|x\|^r \mu_k(dx) \leq c_n a_k^k \quad (k = 1, 2, \ldots), \]
which proves the convergence of series (4.10). This completes the proof of the theorem.

In particular, from Theorem 4 for \( p < 1 \) and every Banach space \( X \) as well as for \( 1 \leq p < r \) and Banach spaces \( X \) of type \( r \) we get the description of \( A_p(X) \).

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