ON SPACES WHOSE PRODUCT WITH EVERY LINDELÖF SPACE IS LINDELÖF

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E. Michael asked whether the product $X^\omega$ is Lindelöf provided that $X$ belongs to $\mathcal{L}$, where $\mathcal{L}$ is the class of all spaces whose Cartesian product with every Lindelöf space is Lindelöf. In this note we present some remarks concerning this problem.

One can view this note as an unsuccessful effort to solve Michael’s problem in the negative way (see [1]–[3] for related problems).

In the sequel, $I$, $Q$, and $N$ stand for the unit interval, the set of rational numbers of the unit interval, and the set of natural numbers, respectively. The symbols $\omega$ and $\omega_1$ denote the first infinite ordinal number and the first uncountable ordinal number, respectively.

If $X$ is a Lindelöf $P$-space, i.e., every $G_\delta$-subset of $X$ is open, then $X$ belongs to $\mathcal{L}$. Noble proved that the product $X^\omega$ is Lindelöf (see [6]), so if Michael’s problem has a negative solution, then one has to look for a counterexample in a class of spaces which are very different from Lindelöf $P$-spaces. The class of Lindelöf spaces in which every point is a $G_\delta$-set is the desired one.

It is easy to prove

**Proposition 1.** Assume that the Continuum Hypothesis holds. Then if $Z \subset I^\Gamma$ and $Z$ belongs to $\mathcal{L}$, then for every countable $S \subset \Gamma$ the space $p_S(Z)$ is $\sigma$-compact, where $p_S$ denotes the projection from $I^\Gamma$ onto $I^S$.

**Proof.** Suppose not. Let $S \subset \Gamma$ be a countable subset of $\Gamma$ such that $p_S(Z)$ is not $\sigma$-compact. Then $P = I^S \setminus p_S(Z)$ is not a $G_\delta$-subset of $I^S$. Let $\{G_\alpha: \alpha < \omega_1\}$ be a strictly decreasing sequence of $G_\delta$-subsets of $I^S$ containing $P$ such that for every open $P \subset U \subset I^S$ there is $\alpha < \omega_1$ such that $G_\alpha \subset U$. For $\alpha < \omega_1$ choose $x_\alpha \in G_\alpha \cap p_S(Z)$ and $x_\alpha \neq x_\beta$ if $\alpha \neq \beta$. Then

$$Y = P \cup \{x_\alpha: \alpha < \omega_1\}$$

with the topology induced by sets of the form $U \cup K$, where $U$ is open in $I^S$ and $K \subset p_S(Z)$ is Lindelöf (see [5], Ex. 1.2). Notice that $Y \times p_S(Z)$ is not
Lindelöf since \( \{(x, x_\alpha): \alpha < \omega_1\} \) is a discrete and closed subset of \( Y \times p_\mathcal{S}(Z) \), and this contradicts the fact that the continuous image of \( \mathcal{L} \) belongs to \( \mathcal{L} \).

The main result of this note is the example which shows that the class of Lindelöf spaces \( X \subset I^{\omega_1} \) such that every point in \( X \) is a \( G_\delta \)-set and, for every \( \alpha < \omega_1 \), the projection of \( X \) onto the first \( \alpha \)-coordinates is \( \sigma \)-compact but \( X \) is not, is not empty.

**Example.** There exists an uncountable space \( X \subset Q^{\omega_1} \) such that

(a) \( p_\alpha(X) \) is countable for every \( \alpha < \omega_1 \), where \( p_\alpha \) is the projection \( p_\alpha: Q^{\omega_1} \to Q^\alpha \);

(b) if \( x \in X \), then \( \{x\} \) is a \( G_\delta \)-subset of \( X \);

(c) for every hereditarily Lindelöf space \( Y \) the Cartesian product \( Y \times X^\alpha \) is Lindelöf.

**Construction of** \( X \). There exists a family \( \{A_\alpha: 1 \leq \alpha < \omega_1\} \) such that

1. \( A_\alpha \) is a countable set consisting of strictly increasing sequences of rational numbers of \( Q \) of length \( \alpha \) for \( 1 \leq \alpha < \omega_1 \);
2. if \( \alpha < \beta < \omega_1 \), then \( p_\alpha(A_\beta) = A_\alpha \);
3. if \( a \in A_\alpha \) for \( 1 \leq \alpha < \omega_1 \), then for every limit ordinal number \( \beta < \alpha \) the numbers

\[
a(\beta) = \lim_{\lambda \to \beta} a(\lambda) \quad \text{and} \quad \sup_{\lambda \to \beta} \{a(\lambda): \lambda < \alpha\}
\]

are rational (see [4], p. 91, the construction of the Aronszajn tree).

Let us attach to \( a \in A_\alpha \), for \( 1 \leq \alpha < \omega_1 \), \( x_\alpha \in Q^{\omega_1} \) such that

\[
x_\alpha(\beta) = \begin{cases} a(\beta) & \text{if } \beta < \alpha, \\ \sup \{a(\lambda): \lambda < \alpha\} & \text{if } \beta \geq \alpha. \end{cases}
\]

Let \( X = \bigcup \{X_\alpha: 1 \leq \alpha < \omega_1\} \), where \( X_\alpha = \{x_\alpha: a \in A_\alpha\} \), be a subspace of \( Q^{\omega_1} \). It is obvious that \( X \) is an uncountable space.

**Proof of** (a). If \( 1 \leq \alpha < \omega_1 \), then from (2) we infer that

\[
p_\alpha(X) \subset A_\alpha \cup p_\alpha \big( \bigcup \{X_\beta: \beta < \alpha\} \big),
\]

so it is countable.

**Proof of** (b). If \( x = x_\alpha \), where \( a \in A_\alpha \), then

\[
\{x \in X: p_{\alpha+2}(x_\alpha) = p_{\alpha+2}(x)\} = \{x_\alpha\}.
\]

By (a), \( p_{\alpha+2}(X) \) is countable, so from the last equality it follows that \( x \) is a \( G_\delta \)-set in \( X \).

**Proof of** (c). Let \( Y \) be a hereditarily Lindelöf space and \( \mathcal{V} \) an open covering of \( Y \times X^\alpha \). Let \( \mathcal{B} = \bigcup \{\mathcal{B}_n: n \in N\} \) be an open basis of \( Q \) such that \( \mathcal{B}_{n+1} \) is a pairwise disjoint open covering of \( Q \) which refines \( \mathcal{B}_n \). We denote
by $S$ the set of finite sequences of natural numbers. For $s \in S$ the symbol $|s|$ stands for the length of $s$ and its elements are denoted by $s_i$ for $i \leq |s|$. If $x = x_a \in X$, where $a \in A$, $x_a(\alpha) \in B \in \mathcal{B}$ and $\alpha < \lambda$, then put

$$F(x, B, \lambda) = \bigcap_{\beta < \omega_1} (P F_\beta \cap X),$$

where

$$F_\beta = \begin{cases} \{x(\beta)\} & \text{if } \beta \leq \alpha, \\ B & \text{if } \beta = \lambda, \\ Q & \text{otherwise.} \end{cases}$$

For $x = (x_1, \ldots, x_n, \ldots) \in X^N$ and $s \in S$, put

$$A(x, s) = \{y \in Y : \text{there are an open neighbourhood } H_y \text{ of } y, \lambda(y) < \omega_1 \text{ and } V \in \mathcal{V} \text{ such that} \}

(y, x) \in H_y \times \bigcap_{i \leq |s|} (F(x_i, B_i, \lambda(y)) \times X \times X \times \ldots \subset V),$$

where $x_i \in X(x_i) = x_i(\alpha) \in B_i \in \mathcal{B}$ for $i \leq |s|$. We can assume, without loss of generality, that $\lambda(y)$ for $y \in A(x, s)$ is as small as possible. Notice that $\{y \in A(x, s) : \lambda(y) < \beta\}$ is an open subset of $A(x, s)$ since it is equal to

$$\bigcup\{H_y : y \in A(x, s) \text{ and } \lambda(y) < \beta\}$$

for $\beta < \omega_1$, and $A(x, s)$ is a Lindelöf space, so

$$\lambda(A(x, s)) = \sup \{\lambda(y) : y \in A(x, s)\} < \omega_1.$$ 

If $A(x, s) = \emptyset$, then put $\lambda(A(x, s)) = 0$. Put

$$\beta_1 = \sup \{\lambda(A(x, s)) : x \in X_1^N \text{ and } s \in S\}.$$ 

Since $X_1^N$ is countable, we have $\beta_1 < \omega_1$. If $\beta_n$ is defined, then put

$$\beta_{n+1} = \sup \{\lambda(A(x, s)) : x \in X_n^N, \alpha \leq \beta_n + 1 \text{ and } s \in S\}.$$ 

Then

$$\beta = \sup \{\beta_n : n \in N\} < \omega_1.$$ 

Put

$$\mathcal{Z} = \{H_y \times \bigcap_{i \leq |s|} (F(x_i, B_i, \lambda(y)) \times X \times X \times \ldots : y \in A(x, s), s \in S, x \in X_a^N \text{ and } \alpha < \beta\}.$$ 

Claim 1. $\mathcal{Z}$ is a covering which refines $\mathcal{V}$.

Proof of the Claim. By the definition, $\mathcal{Z}$ refines $\mathcal{V}$, so it is enough
to show that $\bigcup \mathcal{Y} = Y \times X^\omega$. If $x = x_\alpha \in X$, then put

$$x' = \begin{cases} x & \text{if } a \in \bigcup \{A_\alpha: \alpha < \beta\}, \\ x_\alpha \mid \beta & \text{otherwise}, \end{cases}$$

where $a \mid \beta$ stands for the projection onto the first $\beta$-coordinates. Let $y \in Y$, $x = (x_1, \ldots, x_n, \ldots) \in X^N$ and $x' = (x'_1, \ldots, x'_n, \ldots)$. There exist $k \in \mathbb{N}$, an open neighbourhood $H$ of $y$, $x'_i \in D_i \subset X$, $\alpha < \lambda_i < \omega_1$, $\alpha < \beta$, $B_i \subseteq B_{x_0}$ for $i \leq k$, and $V \in \mathcal{V}$ such that

$$H \times D_1 \times D_2 \times \ldots \times D_k \times X \times X \times \ldots \subseteq V,$$

where

$$D_i = \left( \bigwedge_{\lambda < \omega_1} D_i(\lambda) \right) \cap X,$$

where

$$D_i(\lambda) = \begin{cases} \{x'_i(\lambda)\} & \text{if } \lambda \leq \alpha, \\ B_i & \text{if } \lambda = \lambda_i, \\ \emptyset & \text{otherwise}. \end{cases}$$

Without loss of generality we can assume that $x'_i(\lambda) \in B_i$ for $\lambda \geq \alpha$ and $i \leq k$ (see (3) and (1)). Let $z_i$ for $i \leq k$ be a point of $\bigcup \{X_\lambda: \lambda \leq \alpha + 1\}$ such that

$$z_i(\alpha) = \begin{cases} x'_i(\lambda) & \text{if } \lambda \leq \alpha, \\ x'_i(\alpha) & \text{if } \lambda > \alpha. \end{cases}$$

Put $z = (z_1, z_2, \ldots, z_k, z_k, \ldots)$. Then

$$(y, z) \in H \times D_1 \times \ldots \times D_k \times X \times X \times \ldots \subseteq V$$

because $\alpha < \lambda_i$ and $x'_i(\lambda) \in B_i$ for $\lambda \geq \alpha$ and $i \leq k$. Consequently, $y \in A(z, s)$. By the definition of $\beta$, we have $\lambda(y) < \beta$ for $i \leq k$ and an open neighbourhood $H_y$ of $y$ such that

$$Z(y, z) = H_y \times \bigwedge_{i \leq k} P F(z_i, B_i, \lambda(y)) \times X \times X \times \ldots \subseteq V'$$

for some $V' \in \mathcal{V}$. Notice that from the fact that $x'_i(\lambda) \in B_i$ for $\lambda \geq \alpha$, the definitions of $F(z_i, B_i, \lambda(y))$ and $z_i$ it follows that $(y, x') \in Z(y, z)$. In order to show that $(y, x) \in Z(y, z)$ it is enough to observe that $x' \mid \beta = x \mid \beta$ and that

$$Z(y, z) = H \times \bigwedge_{i \leq k} P_{\beta}^{-1} P_{\beta} \left( F(z(i), B_i, \lambda(y)) \right) \times X \times X \times \ldots$$

The proof will be completed if we show that

**Claim 2.** $\mathcal{Y}$ has a countable subcover.

Let us denote by $p_Y$ and $p_{X^k}$ projections of $Y \times X^N$ onto $Y$ and $X^N$, respectively. Notice that the set

$$\mathcal{P} = \{p_{X^N}(Z): Z \in \mathcal{Y}\}$$
is countable. Put
\[ \mathcal{Z}(P) = \{ Z \in \mathcal{Z} : p_{x \mathbb{N}}(Z) = P \} \quad \text{for } P \in \mathcal{P} \]
and
\[ \mathcal{G}(P) = \{ H : \text{there is } Z \in \mathcal{Z}(P) \text{ such that } p_Y(Z) = H \}. \]
Since \( Y \) is a hereditarily Lindelöf space, there is a countable subfamily \( \mathcal{H}(P) \) of \( \mathcal{G}(P) \) such that \( \bigcup \mathcal{H}(P) = \bigcup \mathcal{G}(P) \). The family
\[ \{ H \times P : H \in \mathcal{H}(P) \text{ and } P \in \mathcal{P} \} \]
is a countable subcover of \( \mathcal{Z} \).

Remark 1. In [1] it was proved that if \( Y \) is a hereditarily Lindelöf space and \( Z \) a Lindelöf \( C \)-scattered space, then \( Y \times Z^\omega \) has the Lindelöf property. Notice that if a Lindelöf \( \sigma \)- (\( C \)-scattered) space \( Z \) is a subspace of \( X \), then \( Z \) is countable. \( X \) does not also contain uncountable separable metric subspaces.

In order to show the first part of the remark it is enough to observe that every compact subset of \( X \) is scattered; otherwise, we could find a countable subset \( S \) of \( \omega_1 \) such that the projection \( p_S(X) \) would be uncountable and that every Lindelöf scattered space in which every point is of \( G_\delta \)-type is countable.

If the second part of the remark does not hold, then we could find a countable subset \( S \) of \( \omega_1 \) such that \( p_S(X) \) would be uncountable.

Let us finish with a result which says that certain subspaces of \( I^{\omega_1} \) with “good projections” do not belong to \( \mathcal{L} \).

Proposition 2. Let \( Z \subset I^{\omega_1} \) be an uncountable space satisfying the following conditions:

(a) \( p_\alpha(Z) \) is countable for \( \alpha < \omega_1 \);
(b) for every \( z \in Z \) there exists \( \alpha_z < \omega_1 \) such that, for every \( \alpha_z < \beta < \omega_1 \),
\[ z(\beta) = z(\alpha_z), \]
and \( z|\alpha_z \) is a strictly increasing sequence;
(c) \( \{ y \in Z : y|\alpha_z = z|\alpha_z \} \) is countable for every \( z \in Z \);
(d) for every \( z \in Z \) and every countable limit ordinal number \( \alpha \),
\[ z(\alpha) = \sup \{ z(\beta) : \beta < \alpha \}. \]

Then \( Z \) does not belong to \( \mathcal{L} \).

Proof. If \( Z \) is not Lindelöf, then there is nothing to prove, so let us assume that \( Z \) has the Lindelöf property. For \( 1 \leq \alpha < \omega_1 \) put
\[ A_\alpha = \{ a \in p_\alpha(Z) : a \text{ is a strictly increasing sequence} \}
and there is no \( z \in Z \) and \( \lambda \leq \alpha \) such that \( \alpha_z = \lambda \) and
\[ z|\lambda = a|\lambda \].

Let \( Y' \) be a subspace of \( I^{\omega_1} \) such that
\[ Y' = \bigcup \{ Y_\alpha : \alpha < \omega_1 \}, \]
where
\[ Y'_a = \{ y_a \in I^{\omega_1} : a \in A_a \} \quad \text{and} \quad y_a|\alpha = a \quad \text{for} \quad a \in A_a, \]
and
\[ y_a(\beta) = \sup \{ a(\lambda) : \lambda < \alpha \} \quad \text{for} \quad \beta \geq \alpha. \]

Notice that \( Y' \) is disjoint with \( Z \). Put \( Y = Y' \cup Z \). The topology on \( Y \) is induced by sets \( U \cup K \), where \( U \) is open in \( I^{\omega_1} \) and \( K \subset Z \).

The product \( Y \times Z \) is not a Lindelöf space. In order to show this it is enough to notice that \( \{(z, z) : z \in Z\} \) is an uncountable discrete closed subset of \( Y \times Z \).

**Claim.** \( Y \) is a Lindelöf space.

We shall split the last claim into two claims:

**Claim (i).** \( Y' \) is a Lindelöf space.

**Claim (ii).** If \( Y' \subset U \) and \( U \) is open in \( Y \), then \( Y \setminus U \) is countable.

**Proof of Claim (i).** The proof is similar to the proof of the Lindelöf property of \( X \). Let \( \mathcal{B} \) be a countable base of \( I \) and \( \mathcal{V} \) an open covering of \( Y' \). For \( y \in Y'_a \), \( \alpha < \omega_1 \), put
\[ \mathcal{A}(y) = \{ B \in \mathcal{B} : y(\alpha) \in B \} \quad \text{and} \quad \exists V \in \mathcal{V} \text{ such that } F(y, B, \beta(y, B)) = \left( \bigwedge_{\lambda < \omega_1} F_\lambda \right) \cap Y' \subset V, \]

where
\[ F_\lambda = \begin{cases} \{ y(\lambda) \} & \text{if } \lambda \leq \alpha, \\ B & \text{if } \lambda = \beta(y, B), \\ I & \text{otherwise}. \end{cases} \]

Since \( Y' \) consists of increasing sequences, \( \mathcal{A}(y) \neq \emptyset \) for every \( y \in Y'_a \) and \( \alpha < \omega_1 \). Put
\[ \beta_1 = \sup \{ \beta(y, B) : y \in Y'_a \text{ and } B \in \mathcal{A}(y) \}. \]

Since \( Y'_1 \) and \( \mathcal{A}(y) \) for \( y \in Y'_1 \) are countable sets, \( \beta_1 < \omega_1 \). If \( \beta_n \) is defined, then put
\[ \beta_{n+1} = \sup \{ \beta(y, B) : y \in \bigcup \{ Y'_{\lambda :} \lambda < \beta_n + 1 \} \text{ and } B \in \mathcal{A}(y) \} \]
and
\[ \beta = \sup \{ \beta_n : n \in \mathbb{N} \}. \]

To complete the proof of the claim it is enough to show that
\[ Y' = \bigcup \{ F(y, B, \beta(y, B)) : y \in \bigcup \{ Y'_{\lambda} : \lambda < \beta \} \text{ and } B \in \mathcal{A}(y) \}. \]
Let $y$ be an element of $Y'_\lambda$ for $\lambda \geq \beta$. Then $y|\beta$ belongs to $A_\beta$. Let $y'$ be a point of $Y'_\beta$ such that $y'|\beta = y|\beta$. There exist $B \in \mathcal{A}$, $x_1, x_2$ and $V \in \mathcal{V}$ such that $y'(\beta) \in B$, $x_1 < \beta < x_2$, and

$$F = \bigcap_{\lambda < \omega_1} F_\lambda \cap Y' \subset V,$$

where

$$F_\lambda = \begin{cases} 
\{y'(\lambda)\} & \text{if } \lambda < x_1,

B & \text{if } \lambda = x_2,

I & \text{otherwise.}
\end{cases}$$

Without loss of generality we can assume that

$$\lim_{\lambda \to x_1} y'(\lambda) \in B.$$

Let $v$ be an element of $Y'_{x_1}$ such that $v|x_1 = y'|x_1$. Then $B \in \mathcal{A}(v)$ and $\beta(v, B) < \beta < x_2$. It is easy to see that $y' \in F(v, B, \beta(v, B))$. Since

$$p_\beta^{-1} p_\beta(F((v, B), \beta(v, B))) = F(v, B, \beta(v, B))$$

and $y'|\beta = y|\beta$,

we have also $y \in F(v, B, \beta(v, B))$.

Proof of Claim (ii). Let $U$ be an open subset of $Y$ such that $Y' \subset U$. There is an open covering $\mathcal{U}$ of $Y'$ in $Y$ such that $H \subset U$ for every $H \in \mathcal{U}$ and

$$H = \bigcap_{\alpha < \omega_1} H_\alpha \cap Y.$$

The family $\mathcal{U}$ has a countable subcover of $Y'$, so there is $\beta < \omega_1$ such that $p_\beta^{-1} p_\beta(Y') \subseteq U$. To complete the proof of the claim it is enough to observe that if $a \in p_\alpha(Z) \setminus p_\beta(Y')$ for $\alpha < \omega_1$, then $\{z \in Z : z|\alpha = a\}$ is countable (see the definition of $A_\alpha$, (b) and (c)), and to apply (a).

Remark 2. We do not know whether $X$ from the Example belongs to $\mathcal{L}$. (P 1333) We think that the answer to this question would be much more interesting than the results of our paper (1).

REFERENCES


(1) The answer is negative (see Problèmes, p. 339).

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