On the length of some curves in the unit sphere

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The purpose of this note is to present simple proofs of the following two theorems.

**Theorem 1.** If \( P_1, \ldots, P_n \) are \( n \) points in the unit sphere \( S \) of the Euclidean \( n \)-dimensional space \( E^n \), \( P_i = (p_{i1}, \ldots, p_{in}) \), \( 1 \leq i \leq n \), and \( p_{ii} = 0 \) for all \( 1 \leq i \leq n \), then

\[
\sum_{i=1}^{n-1} q(P_i, P_{i+1}) \geq \pi/2,
\]

where \( q(x, y) \) is the length of the (shortest) great circle’s arc joining the points \( x \) and \( y \) of \( S \).

**Theorem 2.** If \( P_1, \ldots, P_n \) are \( n \) points in the unit sphere \( S \) of \( E^n \), \( P_i = (p_{i1}, \ldots, p_{in}) \), \( 1 \leq i \leq n \), and \( p_{ii} = 0 \) for all \( 1 \leq i \leq n \), then

\[
\sum_{i=1}^{n} q(P_i, P_{i+1}) \geq \pi \quad \text{(where \( P_{n+1} = P_1 \)).}
\]

These theorems were mentioned by Nehari [2], and were proven analytically by Lasota and Olech [1] (Theorem 1) and Schwartz [4] (Theorems 1 and 2). Schwartz observed that Theorem 2 implies Theorem 1, by simply going along the arc twice, in opposite directions, so as to get a closed arc. We have independently obtained the following essentially different and considerably simpler geometric proofs.

Let \( T_j : E^n \to E^n \) be defined by \( T_j(x_1, \ldots, x_j, \ldots, x_n) = (x_1, \ldots, -x_j, \ldots, x_n) \), for all \( 1 \leq j < n \); \( T_j \) is the reflection of \( E^n \) in the hyperplane \( H_j \), given by the equation \( x_j = 0 \); clearly \( T_j(y) = y \) if and only if \( y \in H_j \); it is well known that each one of the \( T_j \) is an isometric transformation, hence we will use this without proving it; obviously \( T_j(S) = S \) for all \( 1 \leq j \leq n \).

We are ready for the

**Proof of Theorem 1.** Since each \( T_j \) is an isometric transformation,

\[
q(P_i, P_{i+1}) = q[T_iT_{i-1} \ldots T_1(P_i), T_iT_{i-1} \ldots T_1(P_{i+1})]
\]

for all \( 1 \leq i \leq n-1 \),
and therefore

\[ \sum_{i=1}^{n-1} q(P_i, P_{i+1}) = \sum_{i=1}^{n-1} q[T_i T_{i-1} \ldots T_1(P_i), T_i T_{i-1} \ldots T_1(P_{i+1})]. \]

Observe that \( T_1(P_1) = P_1 \), and that for all \( 1 \leq i \leq n-1 \), \( T_{i+1}[T_i T_{i-1} \ldots T_1(P_{i+1})] = T_i T_{i-1} \ldots T_1(P_{i+1}) \), since this last point is in \( H_{i+1} \). In addition, \( T_{n-1} T_{n-2} \ldots T_1(P_n) = -P_n \) since each one of the first \( n-1 \) coordinates of \( P_n \) has been multiplied exactly once by \(-1\), and \( p_{nn} = 0 \).

It therefore follows that the right-hand side of (2) is the length \( l \) of an arc in \( S \), joining \( P_1 \) to \(-P_n\); \( l \) is equal, by (2), to the length of the given arc \( P_1 P_2 \ldots P_n \) in \( S \), joining \( P_1 \) to \( P_n \). Therefore \( 2l \) is the length of an arc in \( S \), joining \( P_n \) to \(-P_n\), and it is well known that an arc in \( S \) that connects two antipodal points has length \( \geq \pi \), therefore \( l \geq \pi/2 \), and the proof is complete.

**Proof of Theorem 2.** As in the previous proof,

\[ q(P_i, P_{i+1}) = q[T_i T_{i-1} \ldots T_2(P_i), T_i T_{i-1} \ldots T_2(P_{i+1})] \]

for all \( 2 \leq i \leq n \),

is true and implies

\[ \sum_{i=1}^{n} q(P_i, P_{i+1}) = \sum_{i=1}^{n} q[T_i T_{i-1} \ldots T_2(P_i), T_i T_{i-1} \ldots T_2(P_{i+1})]. \]

Here the first term in the right-hand side of (2') is \( q(P_1, P_2) \), and \( T_i[T_{i-1} \ldots T_2(P_i)] = T_{i-1} T_{i-2} \ldots T_2(P_i) \), for all \( 2 \leq i \leq n \); in addition, \( T_n T_{n-1} \ldots T_2(P_{n+1}) = T_n T_{n-1} \ldots T_2(P_1) = -P_1 \), since \( P_{n+1} = P_1 \) and it has \( p_{11} = 0 \).

Therefore the left-hand side of (2') is equal, by (2'), to the length of an arc in \( S \) joining \( P_1 \) to \(-P_1\), which is (again) \( \geq \pi \); this completes the proof.

Remark. The idea of the proofs is to keep the first \( i \) parts of the arc while replacing the rest of the arc by the \( T_i \)-image of that rest, doing it successively for \( i = 1, i = 2, \ldots, i = n-1 \) (with a slight variation for the other proof).

Observe that \( n \) applications of Theorem 1 yield a result, weaker than that of Theorem 2; namely with \( \pi \) being replaced by \( \frac{n}{2(n-1)} \pi \).

Our proofs can easily be applied to the following:

**Corollary 1.** If a path in \( S \) contains at least one point on every \( H_j \), for all \( 1 \leq j \leq n \), then it is of length \( \geq \pi/2 \); if, in addition, it is closed, then it is of length \( \geq \pi \).
Corollary 2. If a path in the boundary $C$ of the unit cube in $E^d$ contains at least one point on every $H_j$, for all $1 \leq j \leq n$, then it is of length $\geq 2$; if, in addition, the path is closed, then it is of length $\geq 4$.

This is a particular case of

Corollary 3. If $P$ is a (centrally symmetric) polytope in $E^d$, such that $T_j(P) = P$ for all $1 \leq j \leq n$ and the minimal length of a path in the boundary $Bd(P)$ of $P$, connecting a point $x$ of $Bd(P)$ to $-x$, is $p$; then the length of a path in $Bd(P)$ that contains at least one point of every $H_j$, for all $1 \leq j \leq n$, is $\geq p/2$; if, in addition, the path is closed, then it is of length $\geq p$.

Furthermore, it need not be assumed in Corollary 3 that $P$ is convex, nor polyhedral; the collection of the metric transformations can be replaced by an appropriate finite set of isometric transformations; we omit the rest for obvious reasons.

Using reflections in a similar way, Shaer and Wetzel proved ([3], Lemma 1) that if $a$ is a closed path that meets each and every $(d-1)$-dimensional face of a hyperbox in $E^d$ of diagonal $p$, than the length of $a$ is at least $2p$.

References


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