Some remarks concerning the paper of S. Gołąb and A. Jakubowicz*

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§ 1. Associate with any point \( P(x^1, \ldots, x^n) \) of a space \( L_n \) the space of its bivectors. This space will be regarded as a (generalized) Klein projective space \( K_N \) \((N = \frac{1}{2}n(n-1)-1)\) and every object of \( K_N \) will be termed a \( K \)-object. The generalized Kronecker Deltas \([1]\) and \([2]\)
\[
\delta_P^\varkappa = \delta_P^{[\varkappa]} \quad \delta_P^\zeta = \delta_{[\varkappa]}^\zeta
\]
project a bivector \( h_\varkappa(h_\zeta) \) of \( L_n \) into a \( K \)-point (\( K \)-plane) the homogeneous coordinates of which are
\[
X_P = (\ast)\delta_P^\varkappa h_\varkappa \quad (X_P = (\ast)\delta_P^\varkappa h_\zeta) \quad (1)
\]
The transformation rule is
\[
X_P' = \Delta_P^{X'} X_P, \quad X_P = \Delta_P^X X_P, \\
\Delta_P^{X'} = \delta_P^{\varkappa} \delta_P^{\zeta} A_{\varkappa} A_{\zeta}.
\]
Thus for instance
\[
R_{P_k} = \delta_P^{\varkappa} R_{\varkappa}, \quad R_{\varkappa} = \delta_P^{\varkappa} R_{P_k}.
\]

§ 2. We choose an arbitrary but fixed index \( \zeta_0 \), and an arbitrary but fixed \( K \)-plane \( X_P \), and require
\[
Q_0 = Q_0, \quad X_{Q_0} = \Delta_P^{Q_0} X_P \neq 0 \quad (2)
\]
This is a restriction imposed on coordinate transformation. Using it, we define the non-homogeneous coordinates \( h_P \) of \( X_P \), by
\[
h_P\delta_Q = X_P/X_{Q_0} \quad (3a)
\]

* See, this fascicle, pp. 161-165.

(1) (\ast) stands for an appropriate numerical factor. In the sequel we shall leave this symbol out. \( X \) stands for the Greek \( \chi \) (hi).
The transformation rule of \( h_P \) is obviously
\[
\left(3b\right) \quad h'_P = \varphi h_P A'_P = \frac{h_P A'_P}{h_R A'_Q},
\]
where
\[
\varphi = \frac{1}{h_R A'_Q}.
\]

§ 3. The \( K \)-connection \( \Gamma_{P}^{S} \) can be obtained by means of the requirement that \( \delta_P^S \) (or \( \delta_P^P \)) are covariant constant
\[
D_z \delta_P^S = \Gamma_a^{1} \delta_P^{ar} + \Gamma_a^{r} \delta_P^{ja} - \Gamma_{P}^{R} \delta_R^S = 0.
\]

Multiplying this equation by \( \delta_P^S \) and taking into account
\[
\delta_R^S \delta_P^S = \delta_R^S
\]
on one obtains
\[
\left(4\right) \quad I_{P}^{S} = 2 \delta_P^S \Gamma_{a}^{r} \delta_P^a.
\]

In order to find the covariant derivative of \( h_P \) we use the following definition:
\[
\left(4a\right) \quad D_z h_P = D_z \frac{X_P}{X_Q} = \frac{\left(D_z X_P\right) X_Q - X_P D_z X_Q}{\left(X_Q\right)^2} = \partial_z h_P - h_R \left(- \Gamma_{Q}^{R} h_P + \Gamma_{P}^{R}\right).
\]

If one introduces the symbols
\[
\Lambda_{P}^{R} = \Gamma_{P}^{R} - \Gamma_{Q}^{R} h_Q,
\]
then
\[
\left(4b\right) \quad D_z h_P = \partial_z h_P - \Lambda_{P}^{R} h_R.
\]

Remark. According to \(3a\) we have
\[
\frac{h_Q}{Q} = 1
\]
and this equation is invariant with respect to coordinate transformations (cf. \(3b\)). Hence, we must have
\[
\frac{D_z h_Q}{Q} = 0
\]
as it follows immediately from \(4a\), \(4b\). In particular
\[
\Lambda_{Q}^{R} = \Gamma_{Q}^{R} - \Gamma_{Q}^{R} h_Q = \Gamma_{Q}^{R} - \Gamma_{Q}^{R} = 0.
\]
§ 4. From

\[ R_{\alpha\beta}{}^\gamma = h_{\alpha\beta} R_{\alpha\beta}{}^\gamma \]

one obtains for symmetric connection \( \Gamma_{\mu}{}^\nu \)

\[ R_{\alpha\beta}{}^\gamma h_{\alpha\beta} = 0 \]  

(15)

and this implies

(6a)

\[ h_{\alpha\beta} = h_{\alpha}^{11} h_{\beta}^{21} \]

(6b)

\[ R_{\alpha\beta}{}^\gamma = h_{\alpha}^{1} V_{\beta}{}^\gamma + h_{\beta}^{2} V_{\alpha}{}^\gamma \]

Similarly from

(7)

\[ V_{\xi} R_{\alpha\beta}{}^\gamma = k_{\xi\alpha} R_{\alpha\beta}{}^\gamma \]

we get for symmetric connection

\[ k_{\xi\alpha} = 0 \]

and this implies

\[ k_{\xi\alpha} = k_{\xi} k_{\alpha} l_{\mu} \]

These results suggest that the holonomy group is probably perfect. If this is so, then there are \( n - 2 \) linearly independent absolute parallel vector fields.

References


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