THE REGULAR COMPONENT
OF A GROUP-VALUED SET FUNCTION

BY

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1. Introduction, notation and terminology. Throughout $G$ denotes a complete Hausdorff topological Abelian group, $\mathcal{R}$ stands for a ring of sets, and $\mathcal{A}$ for a subfamily of $\mathcal{R}$ closed under finite unions. It is a consequence of a result due to Traynor [5] that every locally exhaustive additive set function $\mu: \mathcal{R} \to G$ can be uniquely decomposed in the form $\mu = \mu_1 + \mu_2$, where $\mu_1, \mu_2: \mathcal{R} \to G$ are locally exhaustive additive set functions, $\mu_1$ is (inner) $\mathcal{R}$-regular, and $\mu_2$ is $\mathcal{R}$-antiregular (see the proof of Theorem 2 below). Earlier related results can be found in [2] and [3]; see also [6], (3.6) (a). The aim of our paper is to give explicit formulae for $\mu_1$ and $\mu_2$ (see (5) and (6) below).

For every $\mathcal{G} \subset \mathcal{R}$ and $A \in \mathcal{A}$ we set

$$\mathcal{G}_A = \{B \in \mathcal{G} | B \subset A\}.$$

For every $A \in \mathcal{R}$ we denote by $\Delta(A)$ the family of all finite partitions of $A$ contained in $\mathcal{R}$, and we define an order relation $\leq$ on $\Delta(A)$ by setting $\mathcal{Z} \leq \mathcal{M}$ if for each $B \in \mathcal{M}$ there exists $C \in \mathcal{Z}$ with $B \subset C$. Clearly, $\Delta(A)$ is directed by $\leq$.

Let $B \in \mathcal{R}_A$ and $\mathcal{Z} \in \Delta(A)$. If $B$ is the union of a subfamily of $\mathcal{Z}$, we write

$$\mathcal{Z}_B = \{C \in \mathcal{Z} | C \subset B\}.$$

Throughout $\mathcal{U}$ denotes the family of all closed symmetric neighbourhoods of 0 in $G$. For $U \in \mathcal{U}$ we put

$$U^{(n)} = U + \ldots + U \ (n \text{ summands}).$$

An additive set function $\mu: \mathcal{R} \to G$ is called

locally exhaustive if $\mu(A_n) \to 0$ whenever $(A_n)$ is a disjoint sequence in $\mathcal{R}_A$ and $A \in \mathcal{R}$;

$\sigma$-additive if

$$\sum_{m=1}^{n} \mu(A_m) \to \mu\left(\bigcup_{m \in \mathbb{N}} A_m\right)$$

whenever $(A_m)$ is a disjoint sequence in $\mathcal{R}$ with $\bigcup_{m \in \mathbb{N}} A_m \in \mathcal{R}$.
\( \mathcal{R}\)-regular if given \( A \in \mathcal{R} \) and \( U \in \mathcal{U} \) there exists \( K \in \mathcal{R}_A \) such that 
\( \mu(\mathcal{R}_{A \setminus K}) \subseteq U \).

We set 
\[ \text{ea}(\mathcal{R}; G) = \{ \mu: \mathcal{R} \to G \mid \mu \text{ is additive and locally exhaustive} \} . \]

In the sequel, \( \mu \) always denotes an element of \( \text{ea}(\mathcal{R}; G) \).

A ring topology \( \mathcal{I} \) on \( \mathcal{R} \) is called an FN-topology if it admits a base of neighbourhoods of \( \emptyset \) consisting of hereditary subfamilies of \( \mathcal{R} \) (see, e.g., [5], 1.3). We say that \( \mathcal{I} \) is \( \mathcal{R}\)-regular provided for every \( A \in \mathcal{R} \) and every \( \mathcal{I} \)-neighbourhood \( W \) of \( \emptyset \) there exists \( K \in \mathcal{R}_A \) with \( \mathcal{R}_{A \setminus K} \subseteq W \).

We denote by \( \mathcal{I}_\mu \) the weakest FN-topology on \( \mathcal{R} \) with respect to which \( \mu \) is continuous. Then \( \mathcal{I}_\mu \) is \( \mathcal{R}\)-regular if and only if \( \mu \) is \( \mathcal{R}\)-regular.

Finally, we say that \( \mu \) is locally \( \mathcal{I}\)-singular, where \( \mathcal{I} \) is an FN-topology on \( \mathcal{R} \), if, given \( A \in \mathcal{R} \), \( U \in \mathcal{U} \) and a \( \mathcal{I} \)-neighbourhood \( W \) of \( \emptyset \), there exists \( B \in \mathcal{R}_A \) with \( \mu(\mathcal{R}_B) \subseteq U \) and \( A \setminus B \in W \) (cf. [5], 1.4, and [6], pp. 472–473). If \( \mathcal{I} = \mathcal{I}_\nu \), where \( \nu \in \text{ea}(\mathcal{R}; G) \), then \( \mu \) is said to be locally \( \nu\)-singular.

2. Results. We start with an essentially known and easy (cf. [1], 1.5.17)

Lemma 1. For every \( A \in \mathcal{R} \) the net \( \{ \mu(K) \mid K \in \mathcal{R}_A \} \), where the index set is directed upwards by inclusion, satisfies the Cauchy condition.

Using Lemma 1, we can define for every \( A \in \mathcal{R} \)
\[ \psi_\mu(A) = \lim \{ \mu(K) \mid K \in \mathcal{R}_A \} . \]

Moreover, we put for every \( \mathcal{Z} \in \Delta(A) \)
\[ \phi_\mu(\mathcal{Z}) = \sum_{Z \in \mathcal{Z}} \psi_\mu(Z) . \]

Lemma 2. Let \( A \in \mathcal{R} \), let \( \mathcal{Z} = \{ Z_1, \ldots, Z_n \} \in \Delta(A) \) and let \( U \in \mathcal{U} \). Then there exist \( K_i \in \mathcal{R}_{Z_i} \), \( i = 1, \ldots, n \), such that for every \( \mathcal{R} = \{ M_1, \ldots, M_m \} \in \Delta(A) \) with \( \mathcal{R} \geq \mathcal{Z} \) we have
\[ \mu(\bigcup_{j=1}^m L_j \setminus \bigcup_{i=1}^n K_i) \subseteq U \] whenever \( L_j \in \mathcal{R}_{M_j} \), \( j = 1, \ldots, m \).

Moreover, \( \phi_\mu(\mathcal{S}) \in U \) whenever \( \mathcal{S} \in \Delta(D) \) and
\[ \mathcal{S} \geq \{ (Z_1 \setminus K_1) \cap D, \ldots, (Z_n \setminus K_n) \cap D \} \] and \( D = \bigcup_{i=1}^n Z_i \setminus K_i \), \( D \in \mathcal{R} \).

Proof. Choose \( V \in \mathcal{U} \) with \( V^{(n)} \subseteq U \) and \( K_i \in \mathcal{R}_{Z_i} \) with
\[ \mu(L) - \mu(K_i) \in V \] whenever \( K_i \subseteq L \in \mathcal{R}_{Z_i} \), \( i = 1, \ldots, n \)
(see Lemma 1). Put \( N(i) = \{ 1 \leq j \leq m \mid M_j \cap Z_i \neq \emptyset \} \). We have
\[ Z_i = \bigcup_{j \in N(i)} M_j \]
and
\[ \mu(\bigcup_{j \in N(i)} L_j \setminus K_i) = \mu((\bigcup_{j \in N(i)} L_j \cup K_i) \setminus K_i) \in V. \]

This yields the first part of the assertion. The second part is an easy consequence of the first one.

**Lemma 3.** For every \( A \in \mathcal{M} \) the net \( \{ \phi_\mu(\mathcal{Z}) \mid \mathcal{Z} \in \mathcal{A}(A) \} \) satisfies the Cauchy condition.

**Proof.** We assume the contrary. Then there exists \( V \in \mathcal{U} \) such that for every \( \mathcal{Z} \in \mathcal{A}(A) \) we can find \( \mathcal{M} \in \mathcal{A}(A) \) with \( \mathcal{M} \supseteq \mathcal{Z} \) and
\[ (\star) \quad \phi_\mu(\mathcal{M}) - \phi_\mu(\mathcal{Z}) \notin V. \]

Choose \( V_0 \in \mathcal{U} \) with \( V_0^{(3)} \subset V \) and \( V_n \in \mathcal{U} \) with \( \sum_{k=1}^n V_k \subset V_0 \). We shall construct, by recursion, \( K_n \in \mathcal{M}_A \) and \( \mathcal{Z}_n \in \mathcal{A}(A \setminus K_n) \) such that for every \( n \in \mathbb{N} \) the following three conditions hold:

i. \( K_n \supseteq K_{n+1} \);

ii. \( \mu(K_n \setminus K_{n+1}) \notin V_0 \);

(iii) \( \phi_\mu(\mathcal{M}) - \phi_\mu(\mathcal{Z}_n) \in \sum_{k=1}^n V_k \) whenever \( \mathcal{M} \in \mathcal{A}(A \setminus K_n) \) and \( \mathcal{M} \supseteq \mathcal{Z}_n \).

Clearly, (i) and (ii) contradict the local exhaustivity of \( \mu \) as \( K_n \subset A \).

By Lemma 1, choose \( K_1 \in \mathcal{M}_A \) with
\[ \mu(\mathcal{M}_A \setminus K_1) - \mu(\mathcal{M}_A \setminus K_1) \subset V_1 \]
and \( \mathcal{Z}_1 \in \mathcal{A}(A \setminus K_1) \) arbitrarily.

Suppose now the construction has been carried out until some \( n \in \mathbb{N} \). Applying \((\star)\) to \( \mathcal{Z}_n \cup \{ K_n \} \), we obtain \( \mathcal{Z} \in \mathcal{A}(A) \) with the following properties:
\[ \mathcal{Z} \supseteq \mathcal{Z}_n \cup \{ K_n \} \quad \text{and} \quad \phi_\mu(\mathcal{Z}) - (\phi_\mu(\mathcal{Z}_n) + \mu(K_n)) \notin V. \]

We have \( \mathcal{Z}_{K_n} = \{ B_1, \ldots, B_m \} \). Choose \( W \in \mathcal{M} \) with \( W + W \subset V_{n+1} \). By Lemma 2, applied to \( \mathcal{Z}_{K_n} \) and \( W \), there exist \( L_i \in \mathcal{M}_{B_i} \), \( i = 1, \ldots, m \), such that

1. \[ \sum_{i=1}^m \psi_\mu(B_i) - \sum_{i=1}^m \mu(L_i) \in W, \]

2. \[ \phi_\mu(\mathcal{Z}) \in W \quad \text{whenever} \quad \mathcal{Z} \supseteq \{ B_1 \setminus L_1, \ldots, B_m \setminus L_m \}. \]

Put
\[ K_{n+1} = \bigcup_{i=1}^m L_i, \]
and
\[ \mathcal{Z}_{n+1} = \mathcal{Z}_n \cup \{ B_1 \setminus L_1, \ldots, B_m \setminus L_m \}. \]
Clearly, \( \mathcal{Z}_{n+1} \in \mathcal{A}(A \setminus K_{n+1}) \) and (i) holds. By the definition of \( B_i \)'s and \( K_{n+1} \), we have

\[
\phi_\mu(\mathcal{Z}) = \phi_\mu(\mathcal{Z}_{A \setminus K_n}) + \sum_{i=1}^{m} \psi_\mu(B_i),
\]

whence, in view of (1),

\[
\phi_\mu(\mathcal{Z}) - \phi_\mu(\mathcal{Z}_{A \setminus K_n}) - \mu(K_{n+1}) \in V_0.
\]

By (iii) and our choice of \( \mathcal{Z} \), we have

\[
\phi_\mu(\mathcal{Z}_{A \setminus K_n}) - \phi_\mu(\mathcal{Z}_n) \in V_0.
\]

It follows that

\[
\phi_\mu(\mathcal{Z}) - (\phi_\mu(\mathcal{Z}_n) + \mu(K_{n+1})) \in V_0 + V_0,
\]

which yields, in view of our choice of \( \mathcal{Z} \), (ii) for \( n+1 \).

In order to check (iii), fix \( \mathcal{M} \in \mathcal{A}(A \setminus K_{n+1}) \) with \( \mathcal{M} \succcurlyeq \mathcal{Z}_{n+1} \). We have \( \mathcal{M}_{A \setminus K_n} \succcurlyeq \mathcal{Z}_n \), so that, by the inductive hypothesis,

\[
\phi_\mu(\mathcal{M}|_{A \setminus K_n}) - \phi_\mu(\mathcal{Z}_n) \in \sum_{k=1}^{n} V_k.
\]

Since

\[
K_n \setminus K_{n+1} = \bigcup_{i=1}^{m} (B_i \setminus L_i),
\]

we get, in view of (2) and \( \mathcal{M}|_{K_n \setminus K_{n+1}} \succcurlyeq \mathcal{Z}_{n+1}|_{K_n \setminus K_{n+1}} \),

\[
\phi_\mu(\mathcal{M}|_{K_n \setminus K_{n+1}}) - \phi_\mu(\mathcal{Z}_{n+1}|{K_n \setminus K_{n+1}}) \in V_{n+1}.
\]

Summing up (3) and (4), we get (iii) for \( n+1 \).

Using Lemma 3, we can define set functions

\[
\mu, \mu_\omega : \mathcal{R} \to G
\]

by the formulae

\[
\mu(A) = \lim \{ \phi_\mu(\mathcal{Z}) \mid \mathcal{Z} \in \mathcal{A}(A) \},
\]

\[
\mu_\omega(A) = \mu(A) - \mu(A)
\]

for every \( A \in \mathcal{R} \).

**Lemma 4.** Let \( A \in \mathcal{R} \) and \( V \in \mathcal{U} \). Then there exists \( K \in \mathcal{R}_A \) such that

\[
\mu(A) - \mu(K) \in V \quad \text{and} \quad \mu_\omega(\mathcal{R}_A \setminus K) \in V.
\]

**Proof.** Choosing \( U \in \mathcal{U} \) with \( U + U \subseteq V \), we find \( \mathcal{Z} \in \mathcal{A}(A) \), \( \mathcal{Z} = \{ Z_1, \ldots, Z_n \} \), satisfying \( \mu(A) - \phi_\mu(\mathcal{Z}) \in U \). Let \( K_i \in \mathcal{R}_{Z_i} \) be given by Lemma 2. Put

\[
K = \bigcup_{i=1}^{n} K_i.
\]
Then, in view of that lemma, $\phi_\mu(3) - \mu(K) \in U$. It follows that $\mu_r(A) - \mu(K) \in V$.

Fix $D \in \mathcal{R}_A \setminus K$ and let $\mathcal{M} \in \Delta(D)$ satisfy

$$\mathcal{M} \supseteq \{ (Z_1 \setminus K_1) \cap D, \ldots, (Z_n \setminus K_n) \cap D \}.$$ 

Then, in view of Lemma 2, $\phi_\mu(\mathcal{M}) \in U$, so that $\mu_r(D) \in U$.

**Theorem 1.** For every $\mu \in \text{ea}(\mathcal{R}; G)$ we have $\mu_r, \mu_{ar} \in \text{ea}(\mathcal{R}; G)$, and $\mu_r$ is $\mathcal{R}$-regular. Moreover, $\mu$ is $\mathcal{R}$-regular if and only if $\mu = \mu_r$.

**Proof.** We first show that $\mu$ is additive. Let $A, B \in \mathcal{R}$ be disjoint. Fix $V \in \mathcal{I}$ and choose $U \in \mathcal{I}$ with $U^{(3)} \subseteq V$. There exists $3_0 \in \Delta(A \cup B)$ such that

$$\mu_r(A \cup B) - \phi_\mu(3) \in U \quad \text{whenever } 3 \in \Delta(A \cup B) \text{ and } 3 \supseteq 3_0.$$ 

Fix $\mathcal{M}_0 \in \Delta(A)$ and $\mathcal{R}_0 \in \Delta(B)$ with $\mathcal{M}_0 \cup \mathcal{R}_0 \supseteq 3_0$ and

$$\mu_r(A) - \phi_\mu(\mathcal{M}_0) \in U \quad \text{and} \quad \mu_r(B) - \phi_\mu(\mathcal{R}_0) \in U.$$ 

It follows that

$$\mu_r(A \cup B) - (\mu_r(A) + \mu_r(B)) \in U^{(3)} \subseteq V.$$ 

Since $V$ is arbitrary, this yields $\mu_r(A \cup B) = \mu_r(A) + \mu_r(B)$.

As $\mu$ is locally exhaustive, it follows easily from Lemma 4 that $\mu_r$ is also locally exhaustive. The $\mathcal{R}$-regularity of $\mu_r$ follows also immediately from Lemma 4.

Finally, suppose $\mu$ is $\mathcal{R}$-regular. Then $\psi_\mu(A) = \mu(A)$ for every $A \in \mathcal{R}$. Hence $\phi_\mu(3) = \mu(A)$ whenever $A \in \mathcal{R}$ and $3 \in \Delta(A)$. This yields $\mu_r(A) = \mu(A)$ for every $A \in \mathcal{R}$.

**Remarks.** 1. If $\mathcal{R}$ is a $\delta$-ring of sets and $\mu \in \text{ea}(\mathcal{R}; G)$ is $\sigma$-additive, then, in the definitions of $\phi_\mu$ and $\mu_r$, one can replace $\Delta(A)$ by the family of all countable partitions of $A$ contained in $\mathcal{R}$. Moreover, $\mu_r$ and $\mu_{ar}$ are $\sigma$-additive.

2. The second assertion of Lemma 4 states, in the terminology of [4], 1.3 (local setting), that $\mu_r$ is locally nearly supported on $\mathcal{R}$. Moreover, if $\mathcal{R}$ is an ideal of $\mathcal{R}$, then $\mu_{ar}(\mathcal{R}) = \{0\}$. Thus, the decomposition considered in this paper is then a local version of Traynor's decomposition ([4], Theorem 1.7).

The next three lemmas will serve us to prove that $\mu_r$ and $\mu_{ar}$ are, indeed, the $\mathcal{R}$-regular and the $\mathcal{R}$-antiregular components of $\mu$ in the sense of Traynor [5].

**Lemma 5.** If $\mu, \nu \in \text{ea}(\mathcal{R}; G)$, then

(a) $\psi_\mu(\mu + \nu) = \psi_\mu + \psi_\nu$;

(b) $(\mu + \nu)_r = \mu_r + \nu_r$, $(\mu + \nu)_{ar} = \mu_{ar} + \nu_{ar}$.

**Lemma 6.** Suppose $\mu_1, \mu_2 \in \text{ea}(\mathcal{R}; G)$ and $\mu = \mu_1 + \mu_2$. If $(\mu_1)_r = \mu_1$ and $(\mu_2)_r = 0$, then $\mu_1 = \mu_r$ and $\mu_2 = \mu_{ar}$.
Proof. By Lemma 5 (b),
\[ \mu_r = (\mu_1 + \mu_2)_r = (\mu_1)_r + (\mu_2)_r = \mu_1. \]

It is worth-while to note the following assertion, even though it will not be used in the sequel.

**Proposition.** If \( \mu \in \text{ea}(\mathcal{R}; G) \), then
(a) \( (\mu_r)_r = \mu_r \), \( (\mu_{ar})_{ar} = \mu_{ar} \);
(b) \( (\mu_r)_{ar} = 0 = (\mu_{ar})_r \).

**Proof.** The first part of (a) follows from Theorem 1. The remaining assertions can be deduced from this and Lemma 5 (b).

**Lemma 7.** If \( \mu \) is locally \( \mu_r \)-singular, then \( \mu_r = 0 \).

**Proof.** Let \( A \in \mathcal{R} \) and \( V \in \mathcal{U} \). Choose \( U \in \mathcal{U} \) with \( U + U \subset V \). By assumption, there exists \( B \in \mathcal{R}_A \) such that
\[ \mu(\mathcal{R}_B) \subset U \quad \text{and} \quad \mu_r(\mathcal{R}_{A \setminus B}) \subset U. \]

Then, obviously, \( \mu_r(\mathcal{R}_B) \subset U \), whence \( \mu_r(A) \in V \). Since \( V \) is arbitrary, we conclude that \( \mu_r(A) = 0 \).

**Theorem 2.** Let \( \mu \in \text{ea}(\mathcal{R}; G) \) and let \( \mathcal{I}_0 \) be the strongest \( \mathcal{R} \)-regular FN-topology on \( \mathcal{R} \). Then \( \mu_r \) is locally \( \mathcal{I}_0 \)-continuous and \( \mu_{ar} \) is locally \( \mathcal{I}_0 \)-singular.

**Proof.** That \( \mathcal{I}_0 \) exists follows from the simple observation that the family of \( \mathcal{R} \)-regular FN-topologies on \( \mathcal{R} \) is closed under arbitrary suprema.

By Traynor's decomposition theorem ([5], 6.3), there exist (uniquely determined) \( \mu_1, \mu_2 \in \text{ea}(\mathcal{R}; G) \) such that \( \mu_1 \) is locally \( \mathcal{I}_0 \)-continuous, \( \mu_2 \) is locally \( \mathcal{I}_0 \)-singular, and \( \mu = \mu_1 + \mu_2 \). Thus, it is enough to show that \( \mu_1 = \mu_r \) or, in view of Lemma 6, that \( (\mu_1)_r = \mu_1 \) and \( (\mu_2)_r = 0 \). The first assertion follows directly from Theorem 1 as \( \mu_1 \) is \( \mathcal{R} \)-regular. The same theorem yields that \( (\mu_2)_r \) is \( \mathcal{R} \)-regular, so that \( \mu_2 \) is locally \( (\mu_2)_r \)-singular. Accordingly, the second assertion is a consequence of Lemma 7.

**Remark.** 3. In the case where \( G = \mathbb{R} \), the additive group of the reals with the usual topology, the decomposition considered in this paper coincides with the Riesz decomposition in the Dedekind complete Riesz space \( \text{ea}(\mathcal{R}; \mathbb{R}) \) with respect to the band of \( \mathcal{R} \)-regular elements of \( \text{ea}(\mathcal{R}; \mathbb{R}) \). Indeed, \( \mu, v \in \text{ea}(\mathcal{R}; \mathbb{R}) \) are orthogonal if and only if \( \mu \) is locally \( v \)-singular.

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