SOME COVER PROPERTIES
OF NONSEPARABLE METRIC SPACES

BY

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1. Introduction. In this paper we investigate metric spaces which admit so-called \( \alpha \)-zero-bases (\( \alpha \) is a cardinal number cofinal to \( \omega \)) — a generalization of zero-bases introduced by A. Lelek in [9], p. 12–23, and further discussed by A. Lelek in [10] and by R. Duda and R. Telgársky in [4]. In our consideration we use the results of A. H. Stone on absolutely Borel and \( \alpha \)-analytic sets ([11], [12]). We generalize here some theorems of A. H. Stone. We also consider some other cover properties of metrizable spaces.

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2. Preliminaries. Metric spaces are denoted by ordered pairs of the form \((X, d)\). If \((X, d)\) is a metric space, then we use the notation:

\[ B(x, \varepsilon) = \{ y \in X : d(x, y) < \varepsilon \} \quad \text{and} \quad \bar{B}(x, \varepsilon) = \{ y \in X : d(x, y) \leq \varepsilon \}. \]

If \( X \) is a topological space, then \( w(X) \) is the weight of \( X \). If \( \beta \) is an ordinal number, then \( \text{cf}(\beta) \) denotes its cofinality. If \( \beta \) is a cardinal number, then \( \beta^+ \) denotes the first cardinal number greater than \( \beta \).

In this paper \( \alpha \) always denotes a cardinal number of the cofinality \( \omega \).

If \( \alpha \) is given, then we assume that a sequence \( \{ \alpha_n : n < \omega \} \) is also given, in such a way that each \( \alpha_n \) is a cardinal number less than \( \alpha \), \( \alpha_n < \alpha_{n+1} \) and \( \sum \alpha_n = \lim \alpha_n = \alpha \).

3. \( \alpha \)-zero-bases. Let \((X, d)\) be a metric space. A family \( \mathcal{A} \) of subsets of \( X \) is called an \( \alpha \)-zero-family if

(i) \( |\mathcal{A}| \leq \alpha \), and

(ii) for each \( \varepsilon > 0 \) we have \( |\{ A \in \mathcal{A} : \text{diam} A > \varepsilon \}| < \alpha \).

If moreover \( \mathcal{A} \) is a covering of \( X \) (resp. a basis for the topology of \( X \)), then we call it an \( \alpha \)-zero-covering (resp. \( \alpha \)-zero-basis) of \((X, d)\). Clearly, \( \mathcal{N}_0 \)-
zero-basis coincides with a zero-basis. If a metric space \((X, d)\) admits an \(\alpha\)-zero-covering (resp. \(\alpha\)-zero-basis) and \(Y \subset X\), then also \((Y, d)\) admits an \(\alpha\)-zero-covering (resp. \(\alpha\)-zero-basis).

We say that a metric space \((X, d)\) is \(\alpha\)-totally bounded if for each \(\varepsilon > 0\) there is a set \(D \subset X\) such that \(|D| < \alpha\) and if \(x \in X\) is any point, then there is \(y \in D\) such that \(d(x, y) < \varepsilon\). In the case \(\alpha = \aleph_0\), we get the notion of totally bounded metric spaces.

**Theorem 1** (a generalization of [4], (5.7), p. 79). If \((X, d)\) is an \(\alpha\)-totally bounded metric space, then it admits an \(\alpha\)-zero-basis.

**Proof.** For each positive integer \(n\) let \(D_n\) be such a subset of \(X\) that \(|D_n| < \alpha\) and for each point \(x \in X\) there is \(y \in D_n\) such that \(d(x, y) < 1/n\). Then the family \(\{B(y, 1/n) : y \in D_n, 0 < n < \omega\}\) is easily seen to be an \(\alpha\)-zero-basis of \((X, d)\).

Lemma 1 below is a generalization of [5], Theorem 4.3.3, p. 334, and the proof of it is a simple modification of the proof given in [5] for the case \(\alpha = \aleph_0\):

**Lemma 1.** Let \(\{(X_n, \varrho_n) : n < \omega\}\) be a family of nonempty metric spaces such that each metric \(\varrho_n\) is bounded by 1. The Cartesian product \(\prod_{n < \omega} (X_n, \varrho_n)\) with the metric \(\varrho\) defined by the formula

\[
\varrho(x, y) = \sum_{n < \omega} 2^{-n} \varrho_n(x_n, y_n)
\]

(where \(x = (x_0, x_1, \ldots)\) and \(y = (y_0, y_1, \ldots)\)) is \(\alpha\)-totally bounded if and only if all spaces \((X_n, \varrho_n)\) are \(\alpha\)-totally bounded.

Let \(I\) denote the closed unit interval \([0, 1]\) of real numbers and let \(\beta\) be any infinite cardinal number. For each \(\gamma < \beta\) put \(I_\gamma = I \times \{\gamma\}\) and let \(R\) be an equivalence relation in the set \(\bigcup \{I_\gamma : \gamma < \beta\}\) the only nondegenerate class of which is \(\{(0, \gamma) : \gamma < \beta\}\). The quotient set \(J_\beta = \bigcup \{I_\gamma : \gamma < \beta\}/R\) can be metrized by the formula

\[
\delta_\beta([x_1, \gamma_1], [x_2, \gamma_2]) = \begin{cases} 
\frac{1}{2}|x_1 - x_2| & \text{if } \gamma_1 = \gamma_2, \\
\frac{1}{2}(x_1 + x_2) & \text{if } \gamma_1 \neq \gamma_2.
\end{cases}
\]

We consider \(J_\beta\) as a topological space, the topology of which is introduced by \(\delta_\beta\). The space \(J_\beta\) is called the **hedgehog of spininess** \(\beta\) (see [5], Example 4.1.5, p. 314–315).

**Lemma 2.** If \(\alpha > \aleph_0\) then \(J_\alpha \times J_\alpha \times \ldots\) can be embedded into \(J_{\alpha_1} \times J_{\alpha_2} \times \ldots\).

**Proof.** It suffices to show that

\((*)\) \(J_\alpha\) can be embedded into \(J_{\alpha_{k_1}} \times J_{\alpha_{k_2}} \times \ldots\) for any \(0 < k_1 < k_2 < \ldots\)

Indeed, if we take any partition of \(\{1, 2, \ldots\}\) into an infinite family \(\{\{k_1^i, k_2^i, \ldots\} : i = 1, 2, \ldots\}\) of infinite sets, then (by \((*)\)) \(J_\alpha \times J_\alpha \times \ldots\) can be
embedded into
\[ \prod \{ J_{a_{k_i}} \times J_{a_{k_2}} \times \ldots : i = 1, 2, \ldots \}. \]

But the last space is homeomorphic to \( J_{a_1} \times J_{a_2} \times \ldots \).

To show (\( \ast \)) we put \( a_{k_0} = a_0 \) and
\[ f_n([x, y]) = \begin{cases} [x, y] & \text{if } \beta < a_{k_n}, \\ [x, a_{k_n - 1}] & \text{if } \beta \geq a_{k_n}, \end{cases} \]
for \( 0 < n < \omega \). It should be clear that mappings \( f_n: J_a \to J_{a_{k_n}} \) are continuous and, moreover, the family \( \{ f_n: 0 < n < \omega \} \) separates points and closed sets ([5], p. 110). By [5], Theorem 2.3.20, p. 114, the mapping \( f: J_a \to J_{a_{k_1}} \times J_{a_{k_2}} \times \ldots, f(t) = (f_1(t), f_2(t), \ldots), \) is an embedding.

Note that \( f(J_a) \) is not closed in \( J_{a_{k_1}} \times J_{a_{k_2}} \times \ldots \). Indeed, \( \{ [1, a_{k_n}]: 0 < n < \omega \} \) is a sequence which does not converge in \( J_a \) and \( \{ f([1, a_{k_n}]): 0 < n < \omega \} \) is a sequence which converges to a point \( ([1, a_0], [1, a_{k_1}], [1, a_{k_2}], \ldots) \) of \( J_{a_{k_1}} \times J_{a_{k_2}} \times \ldots \).

**Problem 1.** Is \( J_a \times J_a \times \ldots \) homeomorphic to \( J_{a_1} \times J_{a_2} \times \ldots \)? (P 1307)

**Theorem 2.** If \( X \) is a metrizable space of the weight \( \leq \alpha > \aleph_0 \), then there is a metric \( d \) such that the space \( (X, d) \) admits an \( \alpha \)-zero-basis. If, moreover, \( X \) is topologically complete, we may choose a metric \( d \) such that \( (X, d) \) is complete and admits an \( \alpha \)-zero-basis.

**Proof.** By [5], Theorem 4.4.9, p. 353, \( X \) is embeddable into \( J_a \times J_a \times \ldots \), so by Lemma 2, we can consider \( X \) as a subset of \( J_{a_1} \times J_{a_2} \times \ldots \). Since \( w(J_{a_n}) < \alpha \), we see that each \( (J_{a_n}, \delta_{a_n}) \) is \( \alpha \)-totally bounded. By Lemma 1, the space \( (J_{a_1} \times J_{a_2} \times \ldots, \delta_a) \), where
\[ \delta_a'(x_1, x_2, \ldots, x'_1, x'_2, \ldots) = \sum_{n=1}^{\infty} 2^{-n} \delta_{a_n}(x_n, x'_n), \]
is also \( \alpha \)-totally bounded, so by Theorem 1, it admits an \( \alpha \)-zero-basis and therefore so does its subspace \( (X, \delta_a) \). To obtain the second statement we repeat the above consideration for the space \( J_{a_1} \times J_{a_2} \times \ldots \times R \times R \times \ldots \) \((R \) is the real line), because, if \( X \) is a topologically complete subspace of a metrizable space \( Y \), then \( X \) is homeomorphic to a closed subset of \( Y \times R \times \ldots \times R \times \ldots \) (see [5], Lemma 4.3.22, p. 341).

**Problem 2.** Does \( R \times R \times \ldots \) admit a complete metric \( d \) such that there exists an \( \aleph_0 \)-zero-basis of \( (R \times R \times \ldots, d) \)? (P 1308)

**Theorem 3.** If a metric space \( (X, d) \) admits an \( \alpha \)-zero-basis, then for any dense subset \( Y \) of \( X \) and any sequence \( \{ \varepsilon_n > 0: n < \omega \} \) such that \( \lim_{n \to \omega} \varepsilon_n = 0 \), the basis \( \mathcal{B} = \{ B(x, \varepsilon_n): x \in Y, n < \omega \} \) contains some \( \alpha \)-zero-basis \( \mathcal{B}' \) of \( (X, d) \).
Moreover, we can represent $\mathcal{B}$ as a union $\bigcup \{\mathcal{B}_n: n < \omega \}$, where $|\mathcal{B}_n| < \alpha$ and $\mathcal{B}_n \subseteq \{B(x, \varepsilon_n): x \in Y\}$.

**Proof.** We may assume that $X$ contains no isolated point (because $w(X) \leq \alpha$ and hence $|[\{x: x \text{ is an isolated point of } X\}] \leq \alpha$) and that $\varepsilon_0 > \varepsilon_1 > \ldots$

Let $\mathcal{A}$ be an $\alpha$-zero-basis of $(X, d)$. Put

$$\mathcal{A}_0 = \{A \in \mathcal{A}: \text{diam } A \geq \varepsilon_0\}$$

and

$$\mathcal{A}_n = \{A \in \mathcal{A}: \varepsilon_{n-1} > \text{diam } A \geq \varepsilon_n\} \quad \text{for } 0 < n < \omega.$$

Since $\mathcal{A}$ is an $\alpha$-zero-basis, we may index elements of $\mathcal{A}_n$ in such a manner that $\mathcal{A}_n = \{A^\beta_n: \beta < \alpha_n\}$ for some cardinal number $\alpha_n < \alpha$. Moreover, for each $n < \omega$ the family $\bigcup \{\mathcal{A}_k: k \geq n\}$ is still a basis for the topology of $X$. Let $x_n^\beta \in A^\beta_n \cap Y$ be any point, for $0 < n < \omega$, $\beta < \alpha_n$, so $A^\beta_n \subset B(x_n^\beta, \varepsilon_{n-1})$. Put

$$\mathcal{B} = \{B(x_n^\beta, \varepsilon_{n-1}): 0 < n < \omega, \beta < \alpha_n\},$$

so $\mathcal{B}$ is an $\alpha$-zero family. We show that $\mathcal{B}$ is a basis. Let us take any $x \in X$ and $\varepsilon > 0$. There is $n < \omega$ such that $\varepsilon_n < \varepsilon/2$ and there is $A \in \bigcup \{\mathcal{A}_k: k > n\}$ such that $x \in A \subset B(x, \varepsilon_n)$. Hence $A = A^\beta_m$ for some $m > n$, $\beta < \alpha_m$. This means that

$$x \in B(x_m^\beta, \varepsilon_{m-1}) \subset B(x, 2\varepsilon_n) \subset B(x, \varepsilon).$$

**Theorem 4.** Let $(X, d)$ be a metric space such that $X = \bigcup \{X_n: n < \omega\}$, where each $(X_n, d)$ admits an $\alpha$-zero-basis. Then $(X, d)$ also admits an $\alpha$-zero-basis.

**Proof.** Let $n < \omega$ be fixed. By Theorem 3, the space $(X_n, d)$ admits an $\alpha$-zero-basis $\mathcal{B}^n$ such that

$$\mathcal{B}^n = \bigcup \{\mathcal{B}^n_k: 0 < k < \omega\},$$

where

$$\mathcal{B}^n_k = \{B_{x_k}(x^\beta_k(\beta), 1/k): \beta < \gamma^k_n\} \quad \text{for some } \gamma^k_n < \alpha.$$

The family $\{B_x(x^\beta_k(\beta), 1/k): k > n, n < \omega, \beta < \gamma^k_n\}$ is an $\alpha$-zero-basis of $(X, d)$.

Theorem 4 is a generalization of [4], (5.2), p. 77.

**Problem 3.** Does $(X, d)$ admit an $\alpha$-zero-basis provided $X = \bigcup \{X_\beta: \beta < \alpha\}$, where each $(X_\beta, d)$ admits an $\alpha$-zero-basis? (P 1309)

**Theorem 5.** Let $(X, d)$ be a complete metric space such that for each $\varepsilon > 0$ there is $\eta > 0$ such that, for each $x \in X$,

(i) the ball $B(x, \varepsilon)$ contains a set $D$, $|D| = \alpha$ and for any $y, y' \in D$, if $y \neq y'$, then $d(y, y') \geq \eta$.

Then $(X, d)$ does not admit an $\alpha$-zero-basis.
Proof. We may assume that \( w(X) = \alpha \). By (i) we get a sequence \( \{e_n : n < \omega \} \) of positive real numbers such that \( e_0 = 1 \) and for each \( n < \omega \) each \( x \in X \) the ball \( B(x, e_n) \) contains a set \( D^n_x \) such that \( |D^n_x| = \alpha \) and for any \( y, y' \in D^n_x \), if \( y \neq y' \), then \( d(y, y') \geq 5e_{n+1} \). Therefore \( e_n \geq 5e_{n+1} \) and \( \lim_{n \to \infty} e_n = 0 \).

Now we inductively define sets \( A_n = \{a_{\beta_0 \ldots \beta_n} \in X : \beta_0, \ldots, \beta_n < \alpha \} \) for each \( n < \omega \). If \( a \) is any fixed point of \( X \), then put \( A_0 = D^0_x \). Suppose that \( A_n \) is already defined for some \( n < \omega \) and put
\[
A_{n+1} = \bigcup \{D^n_x : x \in A_n \},
\]
where
\[
D^n_{\beta_0 \ldots \beta_n} = \{a_{\beta_0 \ldots \beta_n} : \beta < \alpha \}
\]
for any \( \beta_0, \ldots, \beta_n < \alpha \).

For any \( n < \omega \), \( \beta_0, \ldots, \beta_n < \alpha \) put
\[
B_{\beta_0 \ldots \beta_n} = \text{cl}\left( \bigcup \{B(a_{\beta_0 \ldots \beta_n \ldots \beta_n + m}, e_{n+m+1}) : m < \omega, \beta_{n+1}, \ldots, \beta_{n+m} < \alpha \} \right).
\]
We have
\[
B_{\beta_0 \ldots \beta_n} \subset B(a_{\beta_0 \ldots \beta_n}, 5e_{n+1}/4)
\]
by the triangle inequality for the metric \( d \). Hence if \( x \in B_{\beta_0 \ldots \beta_n}, y \in B_{\beta'_0 \ldots \beta'_n} \) and \( \beta_m \neq \beta'_m \) for some \( m \), \( 0 \leq m \leq n \), then
\[
d(x, y) \geq (5 - 2 \cdot 2^{m+1})e_{m+1} > 2e_{m+1} \geq 2e_{n+1}.
\]

In this way we have obtained the following statement:
there is no ball \( B(z, e_{n+1}), z \in X \), such that both

(ii) \( B_{\beta_0 \ldots \beta_n} \cap B(z, e_{n+1}) \neq \emptyset \) and \( B_{\beta'_0 \ldots \beta'_n} \cap B(z, e_{n+1}) \neq \emptyset \) for any \( n < \omega \), \( \beta_0, \beta'_0, \ldots, \beta_n, \beta'_n < \alpha \), \( \beta_m \neq \beta'_m \) for some \( 0 \leq m \leq n \).

Let us suppose that the basis \( \mathcal{B} = \{B(x, e_n) : x \in X, 0 < n < \omega \} \) of \( X \) contains some \( \alpha \)-zero-basis \( \mathcal{B}' \) of \( (X, d) \). We have \( \mathcal{B}' = \bigcup \{\mathcal{B}_n : 0 < n < \omega \} \), where \( \mathcal{B}_n = \{B(x, e_n) : x \in X \} \) and hence \( |\mathcal{B}_n| < \alpha \). Therefore by (ii) there is \( \gamma_0 < \alpha \) such that \( B_{\gamma_0} \cap \bigcup \mathcal{B}_1 = \emptyset \). Let us suppose that we have already defined all \( \gamma_0, \ldots, \gamma_n \) for some \( n < \omega \). By (ii) there is \( \gamma_{n+1} < \alpha \) such that
\[
B_{\gamma_0 \ldots \gamma_n \gamma_{n+1}} \cap \bigcup \mathcal{B}_{n+2} = \emptyset.
\]

We have \( B_{\gamma_0} \supset B_{\gamma_0 \gamma_1} \supset \ldots ; B_{\gamma_0 \ldots \gamma_n} \) is closed for \( n < \omega \), and
\[
\lim_{n \to \infty} (\text{diam } B_{\gamma_0 \ldots \gamma_n}) \leq \lim_{n \to \infty} e_{n+1} 5/4 = 0,
\]
so by the completeness of \( (X, d) \), the intersection \( \bigcap \{B_{\gamma_0 \ldots \gamma_n} : n < \omega \} \) is a single point \( x_0 \). Since \( x_0 \in B_{\gamma_0 \ldots \gamma_n} \), we see that \( x_0 \notin \bigcup \mathcal{B}_{n+1} \) for each \( n < \omega \), so \( x_0 \notin \bigcup \mathcal{B}' \). This means that \( \mathcal{B}' \) is not even a covering of \( X \). By Theorem 3 \( (X, d) \) does not admit an \( \alpha \)-zero-basis.
Corollary 1. Let $X$ be a Banach space and let $d$ be a metric on $X$ induced by its norm. Then $\dim X < \infty$ if and only if $(X, d)$ admits a zero-basis.

Proof. If $\dim X < \infty$, then $X$ is locally compact and separable. Thus $X$ is $\sigma$-compact, so by [4], (5.8), p. 79, $(X, d)$ admits a zero-basis.

If $\dim X = \infty$, then $X$ is not locally compact at any point, so if $x \in X$, then there is no totally bounded neighbourhood of $x$ (since $(X, d)$ is complete). This shows that $(X, d)$ fulfils the assumptions of Theorem 5 (because each Banach space is metrically homogeneous).

Remark 1. Corollary 1 can be obtained with the use of [3]. Theorem on p. 143, [3]. Remarks on p. 145, and the fact that each open $\aleph_0$-zero-covering of a metric space contains a locally finite subcovering ([10], p. 211).

Example. Let $D_\beta$ denote a discrete space of the cardinality $\beta \geq \aleph_0$. Then $(B_\beta, d_\beta)$ is a metric space, where

$$B_\beta = \{(d_0, d_1, \ldots); d_n \in D_\beta, n < \omega\}$$

and

$$d_\beta((d_0, d_1, \ldots), (d'_0, d'_1, \ldots)) = (\min \{n: d_n \neq d'_n\} + 1)^{-1}$$

for distinct points of $B_\beta$. It is easy to verify that $B_\beta$ is homeomorphic to the countable product of $D_\beta$ with itself and that $(B_\beta, d_\beta)$ is a complete space. For the properties of $(B_\beta, d_\beta)$ see [11], p. 5–8.

(a) From Theorem 5 it follows that $(B_\alpha, d_\alpha)$ does not admit an $\alpha$-zero-basis.

(b) Suppose that $\alpha > \aleph_0$. By [11], 2.4 (1), p. 7, the product space $C_\alpha = \prod \{D_\alpha^n; n < \omega\}$ with the metric $\sigma_\alpha$ is a complete metric space homeomorphic to $B_\alpha$, where

$$\sigma_\alpha((d_0, d_1, \ldots), (d'_0, d'_1, \ldots)) = (\min \{n: d_n \neq d'_n\} + 1)^{-1}$$

for distinct points of $C_\alpha$. The family

$$\{(d_0, \ldots, d_n, d_{n+1}, \ldots); d_{n+1} \in D_{\alpha^{n+1}}, d_{n+2} \in D_{\alpha^{n+2}}, \ldots; n < \omega, d_0 \in D_{\alpha_0}, \ldots, d_n \in D_{\alpha_n}\}$$

is an $\alpha$-zero-basis of $(C_\alpha, \sigma_\alpha)$.

Corollary 2. If a metrizable space $X$ contains a closed subset $Y$ homeomorphic to $B_\alpha$, then there is a metric $d$ on $X$ such that $(X, d)$ does not admit an $\alpha$-zero-basis. If $X$ is topologically complete, we may assume that $(X, d)$ is complete.

Proof. It follows from the part (a) of our Example and [6] (see also [5], Exercise 4.5.20 (c), p. 369). The second statement follows in the same way from Example and [2] (see also [5], Exercise 4.5.20 (f), p. 369).

4. $\alpha$-nucleus. Suppose that $\alpha > \aleph_0$ and let $X$ be a metrizable space. We say that
(i) \( X \) is \( \alpha \)-scarce, if \( X = \bigcup \{ X_n : n < \omega \} \), where \( w(X_n) < \alpha \);

(ii) \( X \) is \( \alpha \)-scarce at the point \( x \), if there is an open set \( U, x \in U \subset X \), such that \( U \) is \( \alpha \)-scarce.

If, moreover, \( w(X) \leq \alpha \), then we define an \( \alpha \)-nucleus \( n_\alpha(X) \) of \( X \) as a set of all points of \( X \) at which \( X \) is not \( \alpha \)-scarce. Therefore \( n_\alpha(X) \) is a closed subset of \( X \) and if \( Y \subset X \), then \( n_\alpha(Y) \subset n_\alpha(X) \).

The notion of an \( \alpha \)-scarce space is quite similar to the notion of a space \( \sigma \)-locally of weight \( < \alpha \) introduced by A. H. Stone ([12], p. 251) and an \( \alpha \)-nucleus of a space is the same as a "nowhere \( \sigma \)lw(\( < \alpha \)) kernel" (if \( w(X) \leq \alpha \) ([12], p. 254).

**Theorem 6.** If \( X \) is a metrizable space such that \( w(X) \leq \alpha > \aleph_0 \), then

(a) if \( X = \bigcup \{ X_\beta : \beta < \alpha \} \) and \( n_\alpha(X_\beta) = \emptyset \), for each \( \beta < \alpha \), then \( X \) is \( \alpha \)-scarce;

(b) if \( \emptyset \neq U \subset n_\alpha(X) \) is open in \( n_\alpha(X) \) and \( U = \bigcup \{ U_\beta : \beta < \alpha \} \), then there is \( \beta_0 < \alpha \) such that \( n_\alpha(U_{\beta_0}) \neq \emptyset \);

(c) \( n_\alpha(X) = \emptyset \) if and only if \( X \) is \( \alpha \)-scarce;

(d) \( n_\alpha(n_\alpha(X)) = n_\alpha(X) \);

(e) if, moreover, \( X \) is topologically complete, then \( X = n_\alpha(X) \) if and only if for each nonempty open subset \( U \) of \( X \) we have \( w(U) = \alpha \).

**Proof.** (a) Since \( w(X_\beta) \leq w(X) \leq \alpha \), we see (by [5], Theorem 1.1.14, p. 34) that there is an open (in \( X_\beta \)) covering \( \{ U_\gamma : \gamma < \alpha \} \) of \( X_\beta \) such that \( U_\beta = \bigcup \{ U_\gamma^n : n < \omega \} \) and \( w(U_\gamma^n) \leq \alpha_n < \alpha \) for each \( \gamma, \beta < \alpha \). Hence

\[
X = \bigcup \{ U_\gamma^n : \gamma, \beta < \alpha; n < \omega \}.
\]

Put

\[
Y_k = \bigcup \{ U_\beta^n : n \leq k; \beta, \gamma < \alpha_k \} \quad \text{for} \quad k < \omega,
\]

so \( w(Y_k) \leq \alpha_k \cdot \alpha \cdot k \) and \( X = \bigcup \{ Y_k : k < \omega \} \) because \( \lim \alpha_n = \alpha \).

(b) If such \( \beta_0 \) does not exist, then by (a) \( U \) is \( \alpha \)-scarce, i.e.,

\[
U = \bigcup \{ U_n : n < \omega \}, \quad \text{where} \quad w(U_n) < \alpha.
\]

Since \( U \) is open in \( n_\alpha(X) \), there is a set \( V \) open in \( X \) such that \( U = n_\alpha(X) \cap V \). Since \( V \setminus U \subset X \setminus n_\alpha(X) \) is open and \( w(V \setminus U) \leq \alpha \), we can find (by [5], Theorem 1.1.14, p. 34) an open (in \( X \)) covering \( \{ V_\beta : \beta < \alpha \} \) of \( V \setminus U \) by \( \alpha \)-scarce sets. Therefore \( n_\alpha(V_\beta) = \emptyset \) for \( \beta < \alpha \), \( n_\alpha(U_n) = \emptyset \) for \( n < \omega \). By (a) \( V \) is an \( \alpha \)-scarce set, so \( U \subset V \subset X \setminus n_\alpha(X) \) — a contradiction.

(c) follows from (a). (d) follows from (b) and (c). (e) is an easy corollary to [12], 2.2 (7), p. 255.

**Remark 2.** The part (a) of Theorem 6 shows that [12], Theorem 3, p. 260, is true for uncountable cardinals of cofinality \( \omega \), without the assumption of Generalized Continuum Hypothesis (compare it with [12], Remark, p. 261).
From Theorem 6 (e) we obtain

**Corollary 3.** (a) \( n_\alpha(B_\alpha) = B_\alpha \) for \( \alpha > \aleph_0 \).
(b) If \( X \) is a Banach space, \( w(X) = \alpha > \aleph_0 \), then \( n_\alpha(X) = X \).

We say that a metrizable space \( X \) is absolutely \( \alpha \)-analytic (\( \alpha > \aleph_0 \)) if \( X \) is a continuous image of \( B_\alpha \). It follows that each absolutely Borel metrizable space of the weight \( \leq \alpha \) is absolutely \( \alpha \)-analytic (see [11], Corollary 3.6, p. 15). For other properties of absolutely \( \alpha \)-analytic spaces see [11].

**Theorem 7.** If \( \alpha > \aleph_0 \) and \( X \) is an absolutely \( \alpha \)-analytic space such that \( n_\alpha(X) \neq \emptyset \), then \( X \) contains a closed subset \( T \) homeomorphic to \( B_\alpha \).

**Proof.** Put \( Y = n_\alpha(X) \), so \( Y \) is an absolutely \( \alpha \)-analytic set (as a closed subset of \( X \)). Therefore there is a continuous map \( f \) from \( B_\alpha \) onto \( Y \). Let \( q \) be a fixed metric on \( Y \). Let \( P \) be the set of all finite nonempty sequences of points of \( D_\alpha \). If \( \sigma = (d_1, \ldots, d_n) \in P \), then put \( l(\sigma) = n \) and

\[
B_{\sigma} = \{(\sigma, d_{n+1}, d_{n+2}, \ldots) = (d_1, \ldots, d_n, d_{n+1}, \ldots) \in B_\alpha: d_{n+1}, d_{n+2}, \ldots \in D_\alpha\}.
\]

Let \( R \) be the set of all finite sequences (including the empty sequence \( \emptyset \)) \( r = (e_1, \ldots, e_n) \) such that \( e_k \in D_{a_k} \), for \( 1 \leq k \leq n \), and let \( Q \) be the set \( \prod \{D_{a_k}: 0 < k < \omega\} \). If \( r \in R \), then let \( l(r) \) be its length.

If \( u \) and \( v \) are sequences such that either \( u, v \in P \) or \( u \in R, v \in R \cup Q \), then \( u \sim v \) denotes the fact that \( v \) is an extension of \( u \) (so, in particular, \( l(u) \leq l(v) \)).

We inductively construct families \( \{e_r > 0: r \in R\} \) and \( \{\sigma_r \in P: r \in R \setminus \{\emptyset\}\} \)

such that:

(i) \( e_r < 1/(l(r)+1) \);
(ii) if \( r_1, r_2 \in R \setminus \{\emptyset\} \), \( r_1 \sim r_2 \), \( l(r_1) < l(r_2) \), then \( \sigma_{r_1} \sim \sigma_{r_2} \) and \( l(\sigma_{r_1}) < l(\sigma_{r_2}) \);
(iii) if \( r, r_1, r_2 \in R \), \( r_1 \neq r_2 \), \( r \sim r_1 \), \( r \sim r_2 \) and \( l(r_1) = l(r_2) = l(r)+1 \), then \( q(f(B(\sigma_{r_1})), f(B(\sigma_{r_2}))) > e_r \);

in particular,

(iii)' \( B(\sigma_{r_1}) \cap B(\sigma_{r_2}) = \emptyset \);
(iv) if \( r \in R \setminus \{\emptyset\} \), then \( n_\alpha(f(B(\sigma_r))) \neq \emptyset \).

Let us suppose that for some \( n < \omega \) we have already defined \( e_r \) and \( \sigma_{r_1} \), for each \( r, r_1 \in R \), \( l(r) = n = l(r_1) - 1 \), such that (i)-(iv) hold. Let \( r_1 \in R \), \( l(r_1) = n+1 \), be fixed and let \( r \in R \) be the unique sequence from \( R \) such that \( r \sim r_1 \) and \( l(r) = n \). By (iv) the set \( A = n_\alpha(f(B(\sigma_{r_1}))) \) is nonempty, so \( w(A) = \alpha \) and hence we can find \( 0 < e_{r_1} < 1/(n+2) \) and a set

\[
D = \{a_d: d \in D_{a_{n+2}}\} \subseteq A
\]
such that \( q(a, a') > 3e_{r_1} \) for \( a, a' \in D, a \neq a' \). Let \( a_d \in D \) be fixed. Since \( f \) is continuous and \( a_d \in f(B(\sigma_{r_1})) \), we can find \( \sigma_{(r_1,a_d)} \in P \) (if \( r_1 = (d_1, \ldots, d_{n+1}) \),
then \((r_1, d) = (d_1, \ldots, d_{n+1}, d)\) such that
\[
\sigma_{(r_1, d)} < \sigma_{r_1}, \quad l(\sigma_{(r_1, d)}) > l(\sigma_{r_1})
\]
and
\[a_d \in f(B(\sigma_{(r_1, d)})) \subset B(a_d, e_{r_1}).\]
Moreover, by Theorem 6 (b), we can take \(\sigma_{(r_1, d)}\) such that
\(n_x\left( f(B(\sigma_{(r_1, d)})) \right) \neq \emptyset\). This finishes our induction.

Put
\[C_k = \bigcup \{B(\sigma_r); \ r \in R, \ l(r) = k\}, \quad \text{for each } 0 < k < \omega,
\]
and
\[C = \bigcap \{C_k; \ 0 < k < \omega\},
\]
so \(C\) is a \(G_\delta\)-subset of \(B_x\). By (ii) and (iii), it is not difficult to see that \(C\) is homeomorphic to \(B_x\) — this follows from [11], Theorem 1, p. 6 (the argument is similar to the consideration in [11], p. 7).

Put \(\mathcal{A}_n = \{\text{cl}(f(B(\sigma_r))): \ r \in R, \ l(r) = n\}\), so by (iii) \(\mathcal{A}_n\) is a discrete family, for each \(0 < n < \omega\). Put \(T_n = \bigcup \mathcal{A}_n\) and \(T = \bigcap \{T_n; \ 0 < n < \omega\}\), so \(T\) is a closed subset of \(X\) (because \(Y\) is closed in \(X\)). It suffices to show that the map \(g: C \to T, \ g(x) = f(x)\) for \(x \in C\), is a homeomorphism.

By (ii) the intersection \(\bigcap \{B(\sigma_r); \ r \in R \setminus \{\emptyset\}, \ r < q\}\) is a single point \(b_q \in C\), for each \(q \in \mathcal{Q}\). Moreover \(C = \{b_q; \ q \in \mathcal{Q}\}\). For each \(q \in \mathcal{Q}\) we have
\[g(b_q) = f(b_q) \in \bigcap \{\text{cl}(f(B(\sigma_r))): \ r \in R \setminus \{\emptyset\}, \ r < q\} \subset T
\]
and by (i)
\[\lim_{l(r) \to \infty} \text{diam}(\text{cl}(f(B(\sigma_r)))) = 0.
\]
This shows that \(g\) is well-defined. Moreover,
\[T \subset \bigcup \{\bigcap \{\text{cl}(f(B(\sigma_r))): \ r \in R \setminus \{\emptyset\}; \ r < q\}; \ q \in \mathcal{Q}\} = g(C),
\]
i.e., \(g\) maps \(C\) onto \(T\). By (iii) \(g\) is a one-to-one map and it is continuous as a restriction of the continuous map \(f\). The inverse map \(g^{-1}: T \to C\) is continuous because the family \(\{B(\sigma_r) \cap C; r \in R \setminus \{\emptyset\}\}\) is a basis of \(C\) and \(g(B(\sigma_r) \cap C) = f(B(\sigma_r) \cap C) \subset f(B(\sigma_r))\) is a closed-open subset of \(T\) (since each family \(\mathcal{A}_k\) is discrete).

Remark 3. Theorem 7 generalizes [11], Theorem 22, p. 37, and [12], Theorem 2, p. 259, which were stated in the case of an absolutely Borel space \(X\). Compare Theorem 7 also with a discussion in [12], Remark on p. 259.

Corollary 4. If \(\alpha > \aleph_0\) and \(X\) is an absolutely \(\alpha\)-analytic space, then the following conditions are equivalent:
(a) \(X\) contains a subset homeomorphic to \(B_x\);
(b) \(n_x(X) \neq \emptyset\);
(c) \(X\) contains a closed subset homeomorphic to \(B_x\).
Proof. Indeed, if \( X \) contains a subset \( Y \) homeomorphic to \( B_\alpha \), then by Corollary 3 (a) we have \( \emptyset \neq n_\alpha(Y) \subset n_\alpha(X) \), so (a) \( \Rightarrow \) (b).

Remark 4. (a) Using Theorem 7 one can show the following generalization of [12], Theorem 5, p. 262: if \( \alpha > \aleph_0 \) and \( X \) is an absolutely \( \alpha \)-analytic space, then \( n_\alpha(X) = \bigcup \{ Y \subset X : Y \text{ is homeomorphic to } B_\alpha \} \) (the proof is similar to the proof of [12], Theorem 5, p. 262).

(b) The implication (a) \( \Rightarrow \) (c) of Corollary 4 is rather surprising but using some ideas of the proof of Theorem 7, the reader can show that if a metrizable space \( X \) contains a subset \( Y \) homeomorphic to \( B_\alpha \), then \( X \) contains a closed subset \( Z \) homeomorphic to \( B_\alpha \) such that \( Z \subset Y \).

5. Hurewicz spaces. Let \( \beta, \gamma \) be cardinal numbers and let \( X \) be a topological space. We say that \( X \) is a \((\beta, \gamma)\)-Hurewicz space, if for any family \( \mathcal{A} = \{ A_\delta : \delta < \gamma \} \) of open coverings of \( X \) there exists a covering \( \mathcal{\mathcal{C}} = \bigcup \{ C_\delta : \delta < \gamma \} \) of \( X \) such that \( C_\delta \subset A_\delta \) and \( |C_\delta| < \beta \) for each \( \delta < \gamma \). Therefore if \( X \) is a \((\beta, \gamma)\)-Hurewicz space and \( \beta' \geq \beta, \gamma' \geq \gamma \), then \( X \) is also a \((\beta', \gamma')\)-Hurewicz space. Moreover, if a topological space \( Y \) is either a continuous image or a closed subset of a \((\beta, \gamma)\)-Hurewicz space, then \( Y \) is again a \((\beta, \gamma)\)-Hurewicz space. It is clear that each compact space is an \((\aleph_0, 1)\)-Hurewicz space, moreover, each \( \sigma \)-compact space is an \((\aleph_0, \aleph_0)\)-Hurewicz space.

\((\aleph_0, \aleph_0)\)-Hurewicz spaces were introduced by W. Hurewicz in [7] \( \text{property } E^* \). Then they were investigated by A. Lelek in [10] \( \text{the name } \) "Hurewicz space" was introduced there. In [10] the reader can find basic properties of \((\aleph_0, \aleph_0)\)-Hurewicz spaces and more complete references than given here.

If \( X \) is a topological space, then we define \( H(X) \) as the least cardinal number \( \gamma \) such that \( X \) is a \((w(X), \gamma)\)-Hurewicz space, so \( 1 \leq H(X) \leq |X| \).

Lemma 3. If \( X \) is a topological space, then each well-ordered by inclusion (either increasing or decreasing) sequence of open subsets of \( X \) has at most \( w(X) \) distinct elements.

Proof. It is a slight modification of [1], IV, §7, Theorem 31 (the Baire–Hausdorff Theorem), p. 161.

Theorem 8. Each topological space \( X \) is a \((2, w(X)^+)\)-Hurewicz space.

Proof. Let \( \{ A_\alpha : \beta < w(X)^+ \} \) be any family of open coverings of \( X \). Let us suppose that for some cardinal number \( \gamma < w(X)^+ \) we have already defined the sequence \( \{ A_\alpha : \beta < \gamma \} \). If \( X \setminus \bigcup \{ A_\alpha : \beta < \gamma \} \neq \emptyset \) then let \( A_\gamma \in \mathcal{A}_\gamma \) be any set and in the opposite case let \( x \) be any fixed point of \( X \setminus \bigcup \{ A_\alpha : \beta < \gamma \} \neq \emptyset \) and let \( A_\gamma \) be any set from \( \mathcal{A}_\gamma \) such that \( x \in A_\gamma \).

Put \( C_\gamma = \bigcup \{ A_\beta : \beta < \gamma \} \) for each \( \gamma < w(X)^+ \). The family \( \{ C_\gamma : \gamma < w(X)^+ \} \) of open sets is well-ordered by inclusion, so by Lemma 3, it has at most \( w(X) \) distinct elements. This means that for some \( \gamma_0 < w(X)^+ \), the set \( X \setminus C_{\gamma_0} \) is empty, i.e., the family \( \{ A_\beta : \beta < \gamma_0 \} \) is a covering of \( X \).
Corollary 5. If X is a topological space, then \( H(X) \leq w(X)^+ \).

Theorem 9. If X is a topological space, \( X = \bigcup \{ X_\varepsilon : \varepsilon < \gamma \} \) and each \( X_\varepsilon \) is a \((\beta, \gamma)\)-Hurewicz space and \( \gamma \geq \aleph_0 \), then also X is a \((\beta, \gamma)\)-Hurewicz space.

Proof. Let \( f : \gamma \times \gamma \to \gamma \) be a fixed one-to-one and onto map \( (f) \) does exist because \( \gamma \geq \aleph_0 \). Let \( \{ \mathcal{A}_\varepsilon : \varepsilon < \gamma \} \) be any family of open coverings of X. Therefore

\[ \mathcal{B} = \{ X_\delta \cap A : A \in \mathcal{A}_\varepsilon : \sigma = f(\delta, \eta) \text{ for } \eta < \gamma \} \]

is a family of open coverings of \( X_\delta \) for each \( \delta < \gamma \). Since \( X_\delta \) is a \((\beta, \gamma)\)-Hurewicz space, we see that there is a covering \( \mathcal{G}^\beta \) of \( X_\delta \) such that

\[ \mathcal{G}^\beta = \bigcup_{\eta < \gamma} \mathcal{G}^\beta_\eta \subset \{ X_\delta \cap A : A \in \mathcal{A}_f(\delta, \eta) \} \quad \text{and} \quad |\mathcal{G}^\beta_\eta| < \beta \quad \text{for} \quad \eta < \gamma. \]

If \( C \in \mathcal{G}^\beta_\eta \) for some \( \delta, \eta < \gamma \), then let \( C' \) be such a fixed element of \( \mathcal{A}_f(A, B) \) that \( C = C' \times X_\delta \). If \( \varepsilon < \gamma \), \( \varepsilon = f(\delta, \eta) \) for some \( \delta, \eta < \gamma \), then put \( \mathcal{A}' \in \mathcal{G}^\beta_\varepsilon = \{ C' \in \mathcal{A}_\varepsilon : C \in \mathcal{G}^\beta_\eta \} \). It is clear that the family \( \bigcup \{ \mathcal{A}' : \varepsilon < \gamma \} \) is an open covering of \( X \) such that \( |\mathcal{A}'| < \beta \) and \( \mathcal{A}' \subset \mathcal{A}_\varepsilon \) for \( \varepsilon < \gamma \), i.e., X is a \((\beta, \gamma)\)-Hurewicz space.

The following Theorem 10 is a generalization of [10], Theorem 1, p. 213.

Theorem 10. Let X be a metrizable space such that \( w(X) \leq \alpha \). The following conditions are equivalent:

(a) X is an \((\alpha, \aleph_0)\)-Hurewicz space;
(b) for every metric \( d \) on X there exists an open \( \alpha \)-zero-covering \( \mathcal{A} \) of X such that \( \text{diam}_d(A) < 1 \), for \( A \in \mathcal{A} \);
(c) for every metric \( d \) on X there exists an \( \alpha \)-zero-basis \( \mathcal{B} \) of \((X, d)\);
(d) there exists a metric \( d \) on X such that every basis \( \mathcal{B} \) of X contains an \( \alpha \)-zero-covering \( \mathcal{A} \) of \((X, d)\);
(e) for every metric \( d \) on X each basis \( \mathcal{B} \) of X contains an \( \alpha \)-zero-covering \( \mathcal{A} \) of \((X, d)\);
(f) for every metric \( d \) on X each basis \( \mathcal{B} \) of X contains an \( \alpha \)-zero-basis \( \mathcal{A}' \) of \((X, d)\).

Proof. It suffices to show that \((f) \Rightarrow (e) \Rightarrow (d) \Rightarrow (a) \Rightarrow (f)\) and \((f) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) \Rightarrow (f)\). The only nontrivial implications are: \((b) \Rightarrow (a)\), \((d) \Rightarrow (a)\) and \((a) \Rightarrow (f)\). The proofs of them are quite similar to those of [10], p. 213–215, given in the case \( \alpha = \aleph_0 \). In the proof of \((b) \Rightarrow (a)\) we should use the following Lemma 4, analogous to [10], Lemma 1.10, p. 213:

Lemma 4. Let X be a paracompact space, \( w(X) \leq \alpha \) and let \( \mathcal{A}_0, \mathcal{A}_1, \ldots \) be open coverings of X. Then there is a pseudo-metric \( p \) on X and open coverings \( \mathcal{C}_0, \mathcal{C}_1, \ldots \) of X such that \( \mathcal{C}_n \) refines \( \mathcal{A}_n \) and for each \( Y \subset X \) satisfying \( \text{diam}_p(Y) < 2^{-n} \), Y intersects only finitely many elements of \( \mathcal{C}_n \), for \( n < \omega \).
Theorem 11. \( H(B_{\aleph_0}) = \aleph_1 \).

Proof. For each \( n < \omega \) put
\[
\mathcal{A}_n = \{(d_0, d_1, \ldots) \in B_{\aleph_0} : d_n + 1, d_n + 2, \ldots \in D_{\aleph_0} : d_0, \ldots, d_n \in D_{\aleph_0}\},
\]
so each \( \mathcal{A}_n \) is an open covering of \( B_{\aleph_0} \). The family \( \{\mathcal{A}_n : n < \omega\} \) shows that \( B_{\aleph_0} \) is not an \((\aleph_0, \aleph_0)\)-Hurewicz space, i.e., \( H(B_{\aleph_0}) > \aleph_0 \).

Theorem 12. Each metrizable space \( X \) such that \( w(X) < \alpha \) and \( n_\alpha(X) \) = \( \emptyset \), for some \( \alpha > \aleph_0 \), is an \((\alpha, \aleph_0)\)-Hurewicz space.

Proof. Since \( X \) is \( \alpha \)-scarce (by Theorem 6(c)), we see that \( X = \bigcup \{X_n : n < \omega\} \), where \( w(X_n) < \alpha \). Therefore each \( X_n \) is an \((\alpha, \aleph_0)\)-Hurewicz space, so by Theorem 9, also \( X \) is such a space.

Theorem 13. Let \( X \) be an absolutely \( \alpha \)-analytic space for some \( \alpha > \aleph_0 \).
(a) \( X \) is an \((\alpha, \aleph_0)\)-Hurewicz space if and only if \( n_\alpha(X) = \emptyset \).
(b) If \( n_\alpha(X) \neq \emptyset \), then \( H(X) = H(B_\alpha) \).

Proof. (a) By Theorem 12 it suffices to show that if \( n_\alpha(X) \neq \emptyset \), then \( X \) is not an \((\alpha, \aleph_0)\)-Hurewicz space. If \( n_\alpha(X) \neq \emptyset \), then by Theorem 7, \( X \) contains a closed subset homeomorphic to \( B_\alpha \), so by Corollary 2, there is a metric \( d \) on \( X \) such that \( (X, d) \) does not admit a \( \alpha \)-zero-basis. By Theorem 10, \( X \) is not an \((\alpha, \aleph_0)\)-Hurewicz space.

(b) Since \( X \) contains a closed subset homeomorphic to \( B_\alpha \), we see that \( H(X) \geq H(B_\alpha) \). But \( X \) is a continuous image of \( B_\alpha \), so \( H(X) \leq H(B_\alpha) \).

Corollary 6. If Generalized Continuum Hypothesis holds and \( X \) is an absolutely \( \alpha \)-analytic space for some \( \alpha > \aleph_0 \), then \( X \) is an \((\alpha, \aleph_0)\)-Hurewicz space if and only if \( |X| \leq \alpha \).

Proof. By [11], Theorem 22, p. 37, if \( |X| > \alpha \), then \( X \) contains a closed subset \( Y \) homeomorphic to \( B_\beta \) for some \( \beta \) such that \( |B_\beta| = \beta^{\aleph_0} > \alpha \). By Generalized Continuum Hypothesis, if \( \beta < \alpha \), then \( \beta^{\aleph_0} \leq \beta^+ < \alpha \). Therefore \( Y \) is homeomorphic to \( B_\alpha \). By Theorem 13, \( X \) is not an \((\alpha, \aleph_0)\)-Hurewicz space.

Remark 5. If we do not assume Generalized Continuum Hypothesis, then it may happen that \( 2^{\aleph_0} = \aleph_{\omega+1} \) (see [8], Theorem 37 (the Easton theorem), p. 63). Therefore \( B_{\aleph_0} \) will be an absolutely \( \aleph_0 \)-analytic space which is an \((\aleph_0, \aleph_0)\)-Hurewicz space and \( |B_{\aleph_0}| > \aleph_0 \).

Problem 4. What is \( H(B_\alpha) \) for \( \alpha > \aleph_0 \)? (P 1310)

References


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