GEOMETRIC STRUCTURE OF POSITIVE BASES IN LINEAR SPACES

1. Introduction. A set \( A \subseteq L \) positively spans a real linear space \( L \) if each element of \( L \) can be represented as a linear combination of elements of \( A \) with positive coefficients. The minimal set which positively spans a space \( L \) is called a positive basis of \( L \) (cf. [1], [4], [7]–[9]). It is well known ([2], [3], [5], [6], [8], [10]) that if \( B \) is a positive basis of a finite-dimensional space \( L \), then

\[
1 + \dim L \leq \card B \leq 2 \dim L,
\]

and for each integer \( k \), \( 1 + \dim L \leq k \leq 2 \dim L \), there exists a positive basis \( B \) of \( L \) such that \( \card B = k \). Moreover, if \( \card B = 1 + \dim L \), then \( B \) is the set of vertices of a simplex containing in its interior the origin. This basis is called a simplicial basis. On the other hand, if \( \card B = 2 \dim L \), then

\[
B = B_1 \cup B_2,
\]

where

\[
B_1 = \{b_1, b_2, \ldots, b_n\}
\]

is a linear basis of \( L \), and

\[
B_2 = \{\beta_1 b_1, \beta_2 b_2, \ldots, \beta_n b_n\},
\]

\[
\beta_i < 0, \quad i = 1, 2, \ldots, n, \quad n = \dim L.
\]

In this case, \( B \) is the so-called maximal basis of \( L \). In the general case, however, the geometric structure of positive bases is not uniquely determined and its description is a rather complicated problem.

The purpose of this paper is to prove a theorem on decomposition of positive bases into the union of disjoint simplices. Such a representation elucidates entirely the geometric structure of positive bases. Moreover, it gives immediately the evaluation (1). Note that this theorem is essentially
stronger than the one formulated in [8]. Additionally, there is given a
characterization of the subsets of $L$, which can be extended to positive bases
in $L$.

Let us first establish the notation and give the indispensable definitions.
We denote by $L^*$ the space of linear functionals defined on $L$ (we assume
that $\dim L < \infty$). For any subset $A$ of $L$ we denote by $\text{lin} A$ ($\text{pos} A$) the set of
all linear combinations of elements of $A$ with real (nonnegative) coefficients.
The set $\text{pos} A$ is a convex cone with vertex at the origin. A set $A$ is linearly
(positively) independent if

$$\text{lin}(A \setminus \{a\}) \neq \text{lin} A \quad (\text{pos}(A \setminus \{a\}) \neq \text{pos} A) \quad \text{for every } a \in A.$$ 

Note that $B$ is a positive basis of $L$ iff $B$ positively spans $L$, and $B$ is
positively independent. By $\text{aff} A$ we denote a carrying flat of the set $A$:

$$\text{aff} A = \{x: x = \sum_{i=1}^{k} \alpha_i a_i, \quad a_i \in A, \quad \sum_{i=1}^{k} \alpha_i = 1\}$$

or

$$\text{aff} A = \text{lin}(A \setminus \{a\}) + a, \quad a \in A.$$ 

The simplex is any set of affinely independent elements of $L$, i.e., the set $D$
such that

$$\text{aff}(D \setminus \{d\}) \neq \text{aff} D \quad \text{for every } d \in D.$$ 

By $\text{conv} A$, $A \subset L$, we denote the convex hull of $A$. We write $f(A) = 0$
($f(A) > 0$), $f \in L^*$, $A \subset L$, if $f(x) = 0$ ($f(x) > 0$) for every $x \in A$. For $f \in L^*$,
$f \neq 0$, we adopt the following notation:

$$H_f = \{x \in L: f(x) = 0\},$$

$$H_f^+ = \{x \in L: f(x) > 0\} \quad \text{and} \quad H_f^- = \{x \in L: f(x) < 0\}.$$ 

Let $S$ be a convex set in $L$. We denote by $\text{relint} S$ (int $S$) the set of points
$x, \ x \in S$, such that

$$\forall y \in S \ \exists \varepsilon > 0 \quad (x + \varepsilon(x - y) \in S)$$

$$(\forall y \in L \ \exists \varepsilon > 0 \quad (x + \varepsilon(x - y) \in S)).$$ 

If $\dim S = \dim L$ ($\dim S = \dim \text{aff} S$), then clearly

$$\text{relint} S = \text{int} S.$$ 

Note that $\text{relint} S$ (relative interior of $S$) is an algebraic notion. If $L$ is,
however, a linear topological space, then $\text{relint} S$ is the interior of $S$ in the
induced topology in the carrying flat of $S$.

A linear combination of elements of $A$ ($A$ is a finite subset of $L$) will be
denoted by $\mathcal{L}(A)$ and, if all coefficients of $\mathcal{L}(A)$ are positive, by $\mathcal{L}^+(A)$.
In this paper we use the following properties of $\text{pos } A$ and $\text{relint } \text{pos } A$:

**Lemma 1.** An element $x$ belongs to $\text{relint } \text{pos } A$ iff for every finite set $M \subset A$ there exists $M_1, M \subset M_1 \subset A$, such that $x = \mathcal{L}^+(M_1)$.

By this lemma we have

1. $x \in \text{relint } \text{pos } A, \ y \in \text{pos } A \Rightarrow x + y \in \text{relint } \text{pos } A,$
2. $x \in \text{relint } \text{pos } A, \ \lambda > 0 \Rightarrow \lambda x \in \text{relint } \text{pos } A.$

**Lemma 2.** If $A \neq \emptyset$ and $B \neq \emptyset$, then

3. $\text{pos } (A \cup B) = \text{pos } A + \text{pos } B,$
4. $\text{relint } \text{pos } (A \cup B) = \text{relint } \text{pos } A + \text{relint } \text{pos } B.$

**Proof.** The property (4) is evident. In order to prove (5) it is sufficient to show that

$$\text{relint } \text{pos } (A \cup B) \subset \text{relint } \text{pos } A + \text{relint } \text{pos } B,$$

because the converse inclusion follows immediately from Lemma 1.

Let $x \in \text{relint } \text{pos } (A \cup B).$ Since $A \neq \emptyset$, there exists $y \in \text{relint } \text{pos } A.$ Clearly, $y \in \text{pos } (A \cup B).$ For a certain $\varepsilon > 0$ we have

$$z = x + \varepsilon(x - y) \in \text{pos } (A \cup B),$$

and hence

$$x = y_1 + z_1, \quad y_1 \in \text{relint } \text{pos } A, \quad z_1 \in \text{pos } (A \cup B).$$

Therefore, by (4) and (2),

$$x = y_2 + z_2, \quad y_2 \in \text{relint } \text{pos } A, \quad z_2 \in \text{pos } B.$$

Analogously,

$$x = y_3 + z_3, \quad y_3 \in \text{pos } A, \quad z_3 \in \text{relint } \text{pos } B,$$

and hence

$$x = \frac{1}{2}(y_2 + z_2) + \frac{1}{2}(y_3 + z_3) = y_4 + z_4,$$

where, by virtue of (2) and (3),

$$y_4 \in \text{relint } \text{pos } A \quad \text{and} \quad z_4 \in \text{relint } \text{pos } B,$$

which completes the proof.

2. Simplicial decomposition of a positive basis. A set $A \subset L$ is called a *positive basis* of $L$ with respect to a subspace $E$ (w.r.t. $E$) if

$$\text{pos } (A \cup E) = L.$$
and for each \( a \in A \)
\[
\text{pos}((A \setminus \{a\}) \cup E) \neq L.
\]
The positive basis of \( L \) w.r.t. \( E = \{0\} \) is the positive basis of \( L \).

**Lemma 3.** Let \( E \) be a proper subspace of \( L \). The following conditions are equivalent:

(i) \( \text{pos}(A \cup E) = L \);
(ii) \( 0 \in \text{int conv}(A \cup E) \);
(iii) for each \( f \in L^* \), \( f \neq 0 \), and \( E \subset H_f \), we have
\[
H_f^+ \cap A \neq \emptyset.
\]

**Proof.** (i) \( \Rightarrow \) (ii). Let \( y \in L \). From (i) it follows that
\[
-y = u + \sum_{i=1}^{k} \lambda_i a_i, \quad u \in E, \quad \lambda_i > 0, \quad a_i \in A.
\]
Taking
\[
\varepsilon = \frac{1}{1+\lambda}, \quad \lambda = \sum_{i=1}^{k} \lambda_i,
\]
we obtain
\[
-\varepsilon y = \frac{1}{1+\lambda} u + \sum_{i=1}^{k} \frac{\lambda_i}{1+\lambda} a_i \in \text{conv}(A \cup E),
\]
and therefore \( 0 \in \text{int conv}(A \cup E) \).

(ii) \( \Rightarrow \) (iii). Let us suppose that \( H_f^+ \cap A = \emptyset \) for a certain \( f \in L^* \), \( f \neq 0 \), \( E \subset H_f \). Then
\[
(6) \quad H_f^+ \cap \text{conv}(A \cup E) = \emptyset.
\]
On the other hand, if (ii) holds, then for \( y \in H_f^+ \) there exists \( \varepsilon > 0 \) such that
\[
-\varepsilon y \in \text{conv}(A \cup E) \quad \text{and} \quad -\varepsilon y \in H_f^+, \quad \text{which contradicts (6)}.
\]

(iii) \( \Rightarrow \) (i). Let us assume that \( \text{pos}(A \cup E) \neq L \). Since \( \text{pos}(A \cup E) \) is a convex cone with the vertex at the origin, by virtue of the Support Theorem there exists \( f \in L^* \), \( f \neq 0 \), such that
\[
\text{pos}(A \cup E) \subset H_f^- \cup H_f.
\]
In this case, \( E \subset H_f \) and \( H_f^+ \cap A \neq \emptyset \), which contradicts (iii).

Now we notice some evident properties of positive bases and positive bases w.r.t. \( E \). In the sequel, we restrict our considerations to the nontrivial case where \( \dim E < \dim L \).

**Statement 1.** If \( B \) is a positive basis of \( L \) w.r.t. \( E \), then the set
\[
B^* = \{ b^* : b^* = b + u_b, \quad b \in B, \ u_b \in E \}
\]
is also a positive basis of \( L \) w.r.t. \( E \) and \( \text{card} B^* = \text{card} B \).
**Statement 2.** Let \( L = L_1 + L_2 \), \( L_1 \cap L_2 = \{0\} \), and let \( B_2 \) be the orthogonal projection of \( B \) onto \( L_2 \). Then the set \( B \) is a positive basis of \( L \) w.r.t. \( L_1 \) iff \( B_2 \) is a positive basis of \( L_2 \).

**Statement 3.** If \( B \) is a positive basis of \( L \) w.r.t. \( E \), then \( B \cap E = \emptyset \) but \( E \cap \text{relint conv } B \neq \emptyset \).

A subset \( B_1 \) of a positive basis \( B \) is called a subbasis of \( B \) if it is a positive basis of the subspace \( \text{lin} B_1 \). If \( B_1 \neq B \), then \( B_1 \) is called a proper subbasis of \( B \).

**Statement 4.** If \( B \) is a positive basis of \( L \) and a set \( B_1 \subset B \) positively spans a subspace \( L_1 \), then \( B_1 \) is a positive basis of \( L_1 \), \( B_1 = L_1 \cap B \), and \( B \setminus B_1 \) is a positive basis of \( L \) w.r.t. \( L_1 \).

Let \( B \) be a positive basis of \( L \). We say that \( C \subset L \) is a critical set of \( B \) if \( \text{pos}(B \setminus \{b\}) \cup C \neq L \) for each \( b \in B \). Elements of \( C \) are called critical vectors of \( B \).

Note that each subset of a critical set is also a critical one. Moreover, the origin is a critical vector for every positive basis. Let us notice some evident properties of critical sets of positive bases.

**Statement 5.** Let \( B \) be a positive basis of \( L \). The following conditions are equivalent:

(i) \( C \) is a critical set for \( B \);

(ii) \( \forall b \in B \exists f \in L^*, f \neq 0 \left( f((B \setminus \{b\}) \cup C) < 0\right) \);

(iii) \( \forall b \in B \left( -\text{pos } C \cap \text{int pos } (B \setminus \{b\}) = \emptyset \right) \);

(iv) \( \forall b \in B \left( -\text{pos } C \cap \text{int pos } ((B \setminus \{b\}) \cup C) = \emptyset \right) \);

(v) \( -\text{pos } C \subset L \setminus \bigcup_{b \in B} \text{int pos } (B \setminus \{b\}) \);

(vi) \( -\text{pos } C \subset L \setminus \bigcup_{b \in B} \text{int pos } ((B \setminus \{b\}) \cup C) \).

**Statement 6.** If \( C \) is a critical set of a positive basis \( B \), then \( \text{pos } C \) and \( \text{cl } C \) are also critical sets of \( B \).

**Statement 7.** If each element of a convex set \( C \) is a critical vector of a positive basis \( B \), then \( C \) is a critical set of this basis.

Let \( C(B) \) denote the set of all critical vectors of a positive basis \( B \) of a space \( L \). According to Statement 5 (v) we obtain

\[
-C(B) = L \setminus \bigcup_{b \in B} \text{int pos } (B \setminus \{b\}).
\]

In the particular case where \( B = \{b_0, b_1, \ldots, b_n\} \) is a simplicial basis of \( L \), \( n = \dim L > 2 \), we have

\[
C(B) = - \bigcup_{i \neq j} \text{pos } (B \setminus \{b_i, b_j\}).
\]

For the maximal basis

\[
C(B) = \{0\}.
\]
Note that if $B = \{b_1, b_2, \ldots, b_m\}$ is a positive basis of $L$, then $B^* = \{\beta_1 b_1, \beta_2 b_2, \ldots, \beta_m b_m\}$, $\beta_i > 0$, is also a positive basis of $L$ and $C(B^*) = C(B)$.

**Statement 8.** Let $B_2$ be a positive basis of $L$ w.r.t. a subspace $L_1$ and let $B_1$ be a positive basis of $L_1$. Then the set $B = B_1 \cup B_2$ is a positive basis of $L$ iff $C = \text{pos } B_1 \cap \text{pos } B_2$ is a critical set of $B_1$.

**Proof.** Suppose that $B = B_1 \cup B_2$ is a positive basis of $L$. If \(\text{pos } (B_1 \setminus \{b\}) \cup C = L_1\) for a certain $b \in B_1$ then, using $B_1 \cap B_2 = \emptyset$, the evident equality $\text{pos } (M \cup C) = \text{pos } M$ for $C \subseteq \text{pos } M$ and (4), we obtain a contradiction:

$$\text{pos } (B \setminus \{b\}) = \text{pos } ((B_1 \setminus \{b\}) \cup C \cup B_2) = L,$$

Now, let $C$ be a critical set of $B_1$ and let

$$\text{pos } (B \setminus \{b\}) = L$$

for a certain $b \in B$. Note that $b \in B_1$, because

$$\text{pos } (L_1 \cup (B_2 \setminus \{b\})) \neq L \text{ for } b \in B_2.$$

Since $L_1 \subseteq \text{pos } (B \setminus \{b\})$, each $x \in L_1$ can be represented as $L^+(M_1) + L^+(M_2)$, where $M_1 \subseteq B_1 \setminus \{b\}$, $M_2 \subseteq B_2$, and $L^+(M_2) \in C$. In this way we get

$$\text{pos } ((B \setminus \{b\}) \cup C) = L,$$

which contradicts the definition of $C$.

**Statement 9.** Every positive basis different from a simplicial one contains a proper subbasis. If $B_1$ is a maximal proper subbasis of $B$, then the set $B \setminus B_1$ is a simplex and

$$\text{lin } B_1 \cap \text{relint conv } (B \setminus B_1) = \{c\},$$

where $c$ is a critical vector of $B_1$.

**Proof.** Since $0 \in \text{int conv } B$ (Lemma 3), there exists a simplex $D \subseteq B$ such that $0 \in \text{relint conv } D$. Since $B$ is not a simplicial basis, the set $D$ is a proper subbasis of $B$.

Let $B_1$ be a maximal proper subbasis of $B$ and let

$$L_1 = \text{lin } B_1 = \text{pos } B_1.$$

Since $B \setminus B_1$ is a positive basis of the space $\text{lin } B$ w.r.t. $L_1$, we have

$$L_1 \cap \text{relint conv } (B \setminus B_1) \neq \emptyset.$$

We will show that this intersection is a one-element set and that the set $B \setminus B_1$ is a simplex. Otherwise, there would exist a simplex $A \subseteq B \setminus B_1$,
$A \neq B \setminus B_1$, such that

$$L_1 \cap \text{relint conv } A \neq \emptyset.$$ 

Let $A$ be a minimal simplex with this property. Hence

$$L_1 \cap \text{relint conv } A = \{c\}$$

and it is evident that $A$ is a positive basis of the space $\text{lin}(L_1 \cup A)$ w.r.t. $L_1$. Therefore, the set $B_1 \cup A$ is a proper subbasis of $B$ which contains $B_1$ as a proper subset. This fact leads to a contradiction with the definition of $B_1$.

Thus we have shown that $B \setminus B_1$ is a simplex and the set $\text{relint conv}(B \setminus B_1)$ cuts $L_1$ at the unique point $c$. From Statement 8 it follows directly that $c$ is a critical vector of $B_1$, and the proof is completed.

Now, let $B = B_1 \cup (B \setminus B_1)$ be a decomposition of the basis $B$ of the space $L$ given by Statement 9. Taking

$$L_1 = \text{pos } B_1, \quad \{c\} = L_1 \cap \text{relint conv } (B \setminus B_1),$$

$$\Delta = (B \setminus B_1) - c, \quad L_2 = \text{aff } \Delta,$$

we obtain a decomposition of the space $L$ into the direct sum a subspace $L_1$ and $L_2$, and a decomposition of the basis $B$ into the union $B_1 \cup (\Delta + c)$, where $B_1$ is a positive basis of $L_1$, $\Delta$ is a simplicial basis of $L_2$, and $c$ is a critical vector of $B_1$.

Note that if $L = L_1 + L_2$, $L_1 \cap L_2 = \{0\}$, $B_1$ is a positive basis of $L_1$, $\Delta$ is a simplicial basis of $L_2$ and $c$ is a critical vector of $B_1$, then $B_1 \cup (\Delta + c)$ is a positive basis of $L$. Hence, using also Statement 9, we obtain

**Theorem 1.** The set $B \subset L$ is a positive basis of $L$ iff $B$ is a simplicial basis of $L$ or $B$ admits the partition

$$B = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_r + c_{r-1}),$$

where $\Delta_1, \ldots, \Delta_r$ are simplicial bases of subspaces $L_1, \ldots, L_r$, $L = L_1 + \ldots + L_r$, $L_i \cap L_j = \{0\}$ for $i \neq j$, $\dim L_i \geq 1$, and $c_j$ ($j = 1, 2, \ldots, r-1$) are critical vectors of bases $B_j$ of the spaces $L_1 + \ldots + L_j$, where

$$B_1 = \Delta_1 \quad \text{and} \quad B_j = \Delta_1 \cup (\Delta_2 + c_1) \cup \ldots \cup (\Delta_j + c_{j-1})$$

for $j = 2, \ldots, r$ ($r \geq 2$).

This theorem characterizes completely the geometric structure of positive bases and provides a method for constructing positive bases in a given linear space.

Let us remark that if

$$\dim L = n, \quad \dim L_i = k_i, \ i = 1, 2, \ldots, r$$
(i.e., \( \text{card} \, A_i = k_i + 1 \)), then by virtue of the obvious equality \( k_1 + \ldots + k_r = n \), we obtain
\[
n + 1 \leq \text{card} \, B = n + r \leq 2n,
\]
because \( 1 \leq r \leq n \).

The equality \( \text{card} \, B = n + 1 \) takes place only for \( r = 1 \), i.e., if \( B \) is a simplicial basis. If \( \text{card} \, B = 2n \), then \( r = n \) and \( k_i = 1 \) for \( i = 1, 2, \ldots, n \). Moreover, by (8) we obtain \( c_j = 0 \) for \( j = 1, 2, \ldots, n - 1 \). Hence in the case \( \text{card} \, B = 2n \) the basis \( B \) has to be of the form
\[
\{b_1, \ldots, b_n, b_{n+1}, \ldots, b_{2n}\},
\]
where \( b_{n+i} = \beta_i b_i, \beta_i < 0, i = 1, 2, \ldots, n \), and \( \{b_1, \ldots, b_n\} \) is a linear basis of the space \( L \).

3. Condition for extension of the set to a positive basis. It is well known that every linearly independent subset of the space \( L \) can be extended to a basis of \( L \). The analogous property for positively independent sets is not so evident. This is easy to see if one considers in \( R^3 \) the set of vertices of a regular pentagon whose carrying plane does not contain the origin.

In this section we formulate necessary and sufficient conditions for the extension of a set to a positive basis in \( L \). Let \( A \) be an arbitrary subset of \( L \) and
\[
Q(A) = \text{relint pos} \, A \setminus \bigcup_{a \in A} \text{relint pos} \, (A \setminus \{a\}).
\]
It is easy to see that if \( A \) is a positive basis of \( L \), then \(-Q(A)\) is the set of critical vectors of \( A \); thus \( 0 \in Q(A) \). We will show that \( A \) admits an extension to a positive basis iff \( Q(A) \neq \emptyset \).

First, let us show a few simple statements.

**Statement 10.** If \( Q(A) \neq \emptyset \), then for an arbitrary nonempty subset \( B \) of \( A \) we have also \( Q(B) \neq \emptyset \).

**Proof.** Suppose that for a certain set \( B \subset A, B \neq A, B \neq \emptyset \), we have \( Q(B) = \emptyset \). Then
\[
\text{relint pos} \, B = \bigcup_{b \in B} \text{relint pos} \, (B \setminus \{b\}).
\]
Let \( x \in \text{relint} \, A \). Then, by Lemma 2,
\[
x = y + z, \quad y \in \text{relint pos} \, B, \quad z \in \text{relint pos} \, (A \setminus B),
\]
and by virtue of (9) there exists \( b \in B \) such that
\[
y \in \text{relint pos} \, (B \setminus \{b\}).
\]
Let \( M \) be a finite subset of \( A \setminus \{b\} \) and let \( M_1 = M \cap B, M_2 = M \setminus M_1 \). Then, using Lemma 1 we have
\[
y = \mathcal{L}^+(M_1 \cup N_1), \quad N_1 \subset B \setminus \{b\},
\]
as well as
\[ z = \mathcal{L}^+ (M \cup N_2), \quad N_2 \subseteq A \setminus B, \]
and hence
\[ x = \mathcal{L}^+ (M \cup N), \quad \text{where } N \subseteq A \setminus \{b\}. \]
Taking into account again Lemma 1 we obtain
\[ x \in \text{relint pos}(A \setminus \{b\}). \]

Due to the arbitrariness of the choice of \( x \) we have \( Q(A) = \emptyset \), which contradicts the assumption.

**Statement 11.** If \( Q(A) \neq \emptyset \), then the set \( A \) is positively independent.

In order to show this statement it is sufficient to notice that if \( \text{pos} A = \text{pos}(A \setminus \{a\}) \) for a certain \( a \in A \), then
\[ \text{relint pos} A = \text{relint pos}(A \setminus \{a\}), \]
which gives \( Q(A) = \emptyset \).

**Statement 12.** \( p \in \text{relint pos} A \) iff
\[ 0 \in \text{relint pos}(A \cup \{-p\}). \]

**Statement 13.** \( \text{lin} A = \text{pos} A \) iff \( 0 \in \text{relint pos} A \).

**Proof.** It is sufficient to notice that if \( x \in \text{lin} A \) and \( x = \mathcal{L}(P), \ P \subseteq A \), then in virtue of \( 0 \in \text{relint pos} A \) and Lemma 1 there exists a subset \( N \) of \( A \) such that
\[ 0 = \mathcal{L}^+(P \cup M \cup N), \]
where \( M \) is an arbitrary subset of \( A \). Thus
\[ x = t \mathcal{L}^+(P \cup M \cup N) + \mathcal{L}(P) = \mathcal{L}^+(P \cup M \cup N) \]
provided that \( t \) is a sufficiently large positive number.

**Statement 14.** If
\[ 0 \in \text{relint pos} A \quad \text{and} \quad 0 \in \text{relint pos}(A \setminus \{a\}) \]
for a certain \( a \in A \), then \( Q(A) = \emptyset \).

**Proof.** It follows from the assumption that for an arbitrary set \( M \subseteq A \setminus \{a\} \) we have
\[ 0 = \mathcal{L}^+(M \cup \{a\} \cup N_1), \quad N_1 \subseteq A \setminus \{a\}, \]
and
\[ 0 = \mathcal{L}^+(M \cup N_1 \cup N_2), \quad N_2 \subseteq A \setminus \{a\}. \]
Thus
\[ a = tL^+ (M \cup N_1 \cup N_2) - L^+ (M \cup N_1) = L^+ (M \cup N_1 \cup N_2), \]

\[ N_1 \cup \bar{N}_2 \subset A \setminus \{a\}, \]

provided that \( t \) is a sufficiently large positive number. This means that \( a \in \text{relint pos} (A \setminus \{a\}) \), and hence \( a \in \text{pos} (A \setminus \{a\}) \). The equality \( Q(A) = \emptyset \) follows now immediately from Statement 11.

**Statement 15.** A set \( A \) is a positive basis of \( \text{lin} A \) iff \( 0 \in Q(A) \).

**Proof.** If \( A \) is a positive basis, then, according to the previous considerations, \( 0 \in Q(A) \). If \( 0 \in Q(A) \), then \( 0 \in \text{relint pos} \ A \) but \( 0 \notin \text{relint pos} (A \setminus \{a\}) \) for each \( a \in A \). By Statement 13 we have \( \text{lin} A = \text{pos} A \) as well as

\[ \text{lin} A \neq \text{pos} (A \setminus \{a\}) \quad \text{for each} \ a \in A. \]

Thus \( A \) is a positive basis of \( \text{lin} A \).

**Statement 16.** If \( 0 \notin Q(A) \), then \( p \in Q(A) \) iff

\[ 0 \in Q(A \cup \{-p\}). \]

**Proof.** If \( p \in Q(A) \), then clearly

\[ p \in \text{relint pos} \ A \quad \text{and} \quad p \notin \text{relint pos} (A \setminus \{a\}) \]

for each \( a \in A \). Hence, by virtue of Statement 12, we have

\[ 0 \in \text{relint pos} (A \cup \{-p\}) \quad \text{and} \quad 0 \notin \text{relint pos} ((A \setminus \{a\}) \cup \{-p\}) \]

for each \( a \in A \). Moreover, since \( 0 \notin Q(A) \) and \( Q(A) \neq \emptyset \), we have also (cf. Statement 13) \( 0 \notin \text{relint pos} \ A \). Hence we obtain (10).

Assume now that (10) holds. Clearly, \( -p \notin A \). By virtue of Statement 12 we have

\[ p \in \text{relint pos} \ A \quad \text{and} \quad p \notin \text{relint pos} (A \setminus \{a\}) \quad \text{for each} \ a \in A, \]

which gives us \( p \in Q(A) \).

**Theorem 2.** A nonempty set \( A \subset L \) can be extended to a positive basis of a space \( L \) iff \( Q(A) \neq \emptyset \).

**Proof.** If \( A \) is a subset of a positive basis \( B \), then \( Q(B) \neq \emptyset \) (cf. Statement 15), and hence \( Q(A) \neq \emptyset \) (by Statement 10). Let now \( Q(A) \neq \emptyset \). It is sufficient to show that \( A \) admits an extension to a positive basis of \( \text{lin} A \). If \( 0 \in Q(A) \), then by Statement 15 the set \( A \) is a positive basis of \( \text{lin} A \). If \( 0 \notin Q(A) \), then there exists \( p \in Q(A), p \neq 0 \), such that the set \( B_1 = A \cup \{-p\} \) is a positive basis of \( \text{lin} A = \text{pos} B_1 \) (cf. Statement 16), which completes the proof.
Remark 1. If \( Q(A) \neq \emptyset \) and \( A \) is a subset of an \( n \)-dimensional space, then \( \text{card } A \leq 2n \).

Remark 2. If \( 0 \notin Q(A) \), i.e., \( A \) is not a positive basis of \( \text{lin } A \), then \( A \cup \{-p\} \) is a positive basis of \( \text{lin } A \) iff \( p \in Q(A) \).

References


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