CONVERGENCE IN THE DUAL OF CERTAIN $K\{M_p\}$-SPACES

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In this note we give two theorems characterizing convergent sequences in the dual of certain $K\{M_p\}$-spaces (see [3], Chapter II for material on $K\{M_p\}$-spaces). Theorem 6 gives a characterization in terms of the usual representation of elements in $K\{M_p\}'$, when $\{M_p\}$ satisfies conditions (M), (N) and (P) (see [3], II. 2.3 and 4.2); this result is somewhat analogous to the convergence criteria for $\mathcal{F}'$ given in Theorem 56, Chapter 3 of [2], and to the convergence criteria for $\mathcal{H}_c'$ given in Theorem 3 of [8]. Theorem 7 gives a characterization in terms of regularizers. There do not seem to be any results analogous to Theorem 7 recorded, even for the case where $K\{M_p\} = \mathcal{F}$.

First, we recall some facts pertinent to $K\{M_p\}$-spaces. Let $\{M_p\}$ be a sequence of extended real-valued functions defined on $\mathbb{R}^n$ such that $1 \leq M_1(x) \leq M_2(x) \leq \ldots$ It is further assumed that at each point $x \in \mathbb{R}^n$ all the $M_p(x)$ are finite or infinite. If $S = \{x: M_p(x) < \infty, p \geq 1\}$, it is assumed that $M_p$ restricted to $S$ is continuous. An infinitely differentiable function $\varphi$ defined on $\mathbb{R}^n$ belongs to $K\{M_p\}$ if

1. $D^a\varphi(x) = 0$ for $x \notin S$ and any multi-index $a$,
2. $M_p D^a\varphi$ is a continuous bounded function on $S$ for $1 \leq p < \infty$ and $0 \leq |a| \leq p$.

The vector space $K\{M_p\}$ is then given the locally convex topology generated by the norms

3. $\|\varphi\|_p = \sup\{M_p(x)|D^a\varphi(x)|: x \in S, |a| \leq p\}$ \hspace{1cm} (1 \leq p < \infty).

We will only consider $K\{M_p\}$-spaces which satisfy three further conditions. The sequence $\{M_p\}$ satisfies the following conditions:

(M) Each $M_p$ is quasi-montonic, i.e., for $|x'_j| \geq |x''_j|$ and $x'_j$ and $x''_j$ having the same sign,

$$M_p(x_1, \ldots, x'_j, \ldots, x_m) \geq C_p M_p(x_1, \ldots, x''_j, \ldots, x_m).$$
(N) For each \( p \), there exists an integer \( p' > p \) such that the quotient 
\( M_p(x) = m_{pp'}(x) \) is summable on \( \mathbb{R}^m \) and \( m_{pp'}(x) \to 0 \) as \( |x| \to \infty \).
(Here it is understood that the quotient \( \infty / \infty \) is 0.)

(P) For \( \varepsilon > 0 \) and \( p \) an integer, there exist \( p' > p \) and \( N \) such that
\( M_p(x) < \varepsilon M_p'(x) \) if \( |x| > N \) or \( M_p(x) > N \).

Many of the familiar test spaces are \( K\{M_p\} \)-spaces which satisfy
conditions (M), (N) and (P).

Example 1. For \( K \subseteq \mathbb{R}^m \) compact, set \( M_p(x) = 1 \) if \( x \in K \) and
\( M_p(x) = \infty \) if \( x \notin K \). Then \( K\{M_p\} = \mathscr{D}_K \) (see [5]), and \( \{M_p\} \) satisfies (M),
(N) and (P).

Example 2. If \( M_p(x) = (1 + |x|)^p \), then \( K\{M_p\} = \mathbb{S} \) is the space
of rapidly decreasing functions [5]. \( \{M_p\} \) is easily seen to satisfy conditions
(M), (N) and (P).

Example 3. If \( M_p(x) = \exp(p \gamma(x)) \), where \( \gamma(x) = (1 + |x|^2)^{1/2} \), then
\( K\{M_p\} \) is the test space \( \mathcal{X}_1 \) of [9]. Again conditions (M), (N) and (P)
are satisfied.

Example 4. Let \( \{r_j\} \) be a sequence such that \( 0 < r_1 < r_2 < \ldots < r \)
and \( r_j \to r \). Set \( M_p(t) = \exp(r_p |t|) \) for \( t \in \mathbb{R} \). In this case \( K\{M_p\} = H_r \) as
in [8], and \( \{M_p\} \) satisfies conditions (M), (N) and (P).

In section II.4.2 of [3] it is shown that the sequence of norms

\[
\|\varphi\|_{p,1} = \sup_{|\alpha| \leq p} \int M_p(x) |D^\alpha \varphi(x)| \, dx \quad (p \geq 1)
\]

(4)

generates the same locally convex topology on \( K\{M_p\} \) as the sequence
\( \{\| \|_p\} \) given in (3). (Here \( \int f(x) \, dx \) denotes the integral of \( f \) over \( S \).) To
obtain our first result we consider another sequence of norms. Note that
since \( m_{pp'} \) in condition (N) is summable over \( \mathbb{R}^m \) and \( m_{pp'}(x) \to 0 \) as \( |x| \to \infty \),
we infer that \( m_{pp'} \in L^2(\mathbb{R}^m) \). If \( \varphi \in K\{M_p\} \) and \( |\alpha| \leq p \), then

\[
M_p(x) |D^\alpha \varphi(x)| \leq m_{pp'}(x) \|\varphi\|_{p'},
\]

where \( p' \) is given by (N), so that \( M_p D^\alpha \varphi \) is in \( L^2(\mathbb{R}^m) \). Thus we may consider
the sequence of norms given by

\[
\|\varphi\|_{p,2} = \sup_{|\alpha| \leq p} \left( \int (M_p(x) |D^\alpha \varphi(x)|^2 \, dx \right)^{1/2} \quad (p \geq 1).
\]

(5)

(Similar \( L^2 \)-type norms are considered in Theorem 7 of I. 3.6 of [4].)
First, we show that the sequence of norms in (5) is equivalent to the
sequence of norms in (3).

**Lemma 5.** The sequence of norms \( \{\| \|_p\} \) is equivalent to the sequence of
norms \( \{\| \|_{p,2}\} \), i.e., the two sequences generate the same locally convex topology
on \( K\{M_p\} \).
Proof. Given $p$ and $\alpha$ with $|\alpha| \leq p$, we have, for $\varphi \in K\{M_p\}$,

$$\int M_p^2(x)|D^\alpha \varphi(x)|^2 \, dx \leq \sup_x M_p^{2'}(x)|D^\alpha \varphi(x)|^2 \int m_{pp'}^2(x) \, dx,$$

where $p'$ is given as in condition (N). Thus there is a constant $C_p > 0$ such that $|\varphi|_{p,2} \leq C_p |\varphi|_{p'}$. On the other hand, there is a constant $A_p$ and a positive integer $q \geq p$ such that $|\varphi|_p \leq A_p |\varphi|_{q,1}$ (see [3], II. 4.2). Let $q'$ correspond to $q$ as in condition (N). Then we have, by the Cauchy-Schwarz inequality,

$$\int M_q(x)|D^\alpha \varphi(x)| \, dx \leq \left( \int m_{qq'}^2(x) \, dx \right)^{1/2} \left( \int M_q^2(x)|D^\alpha \varphi(x)|^2 \, dx \right)^{1/2} \quad \text{for } |\alpha| \leq q.$$

Thus there is a constant $B_p$ such that $|\varphi|_p \leq A_p |\varphi|_{q,1} \leq B_p |\varphi|_{q',2}$ and the lemma follows.

Remark. The equivalence of $\{||_p\}$ and $\{||_{p,2}\}$ is proved in Theorem 7 of I.3.6 of [4] with some additional assumptions on the $\{M_p\}$. (See equation (10) of I.3.6 in [4]; in particular, it is assumed that the $M_p$ are infinitely differentiable.) From the lemma, these additional assumptions are not necessary.

We now give the first characterization of sequential convergence in $K\{M_p\}'$.

**Theorem 6.** Let $\{M_p\}$ satisfy conditions (M), (N) and (P). The following conditions are equivalent:

(i) $T_n \to 0$ weakly (strongly) in $K\{M_p\}'$ (see [3], I.6.4);

(ii) there exist a positive integer $p$ and, for each multi-index $\alpha$ with $|\alpha| \leq p$, a sequence $(f_{\alpha,n})_{n=1}^\infty \subseteq L^2(S)$ such that

$$T_n = \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^\alpha (M_p f_{\alpha,n}) \quad \text{and} \quad f_{\alpha,n} \to 0 \quad \text{in } L^2(S).$$

Proof. Suppose $T_n \to 0$ weakly in $K\{M_p\}'$. By I.6.4 of [3], there is a positive integer $p$ such that

$$\sup \{ |\langle T_n, \varphi \rangle| : \varphi \in K\{M_p\}, |\varphi|_{p,2} \leq 1 \} \to 0 \quad \text{as } n \to \infty. \quad (6)$$

In particular, there is a constant $B > 0$ such that $|\langle T_n, \varphi \rangle| \leq B |\varphi|_{p,2}$ for $\varphi \in K\{M_p\}$. Let $\Gamma'$ be the direct sum of a finite number (equal to the number of multi-indices $\alpha$ such that $|\alpha| \leq p$) of copies of $L^2(S)$ and equip $\Gamma'$ with the norm

$$\|\{f_{\alpha}\}_{|\alpha| \leq p}\| = \sup_{|\alpha| \leq p} \|f_{\alpha}\|_2, \quad \text{where} \quad \|f_{\alpha}\|_2 = \left( \int |f_{\alpha}(x)|^2 \, dx \right)^{1/2}.$$

Define a map $\theta$ from $K\{M_p\}$ into $\Gamma$ by $\theta : \varphi \to \{M_p D^\alpha \varphi\}_{|\alpha| \leq p}$, and note that $\theta$ is one-one. Let $\Lambda$ be the image of $K\{M_p\}$ under $\theta$, $\Lambda$ the closure
of $\Lambda$ in $\Gamma$, and $\Lambda^\perp$ the orthogonal complement of $\Lambda$ in $\Gamma$. For each $n$, define a linear functional $L_n$ on $\Lambda$ by $\langle L_n, \theta(\varphi) \rangle = \langle T_n, \varphi \rangle$. Since
$$|\langle L_n, \theta(\varphi) \rangle| \leq B \|\varphi\|_{p,2} = B \|\theta(\varphi)\|,$$
$L_n$ is continuous. We extend $L_n$ to $\bar{\Lambda}$ by continuity, and then to $\Gamma$ by setting $\langle L_n, g \rangle = 0$ for $g \in \bar{\Lambda}^\perp$. This extension, which we continue to denote by $L_n$, has the same norm as $L_n$ over $\Lambda$. But, by (6), $\|L_n\| \to 0$ in $\Gamma'$ as $n \to \infty$. By the Riesz Representation Theorem, for each $n$, there exist functions $\{f_{a,n}: |a| \leq p\} \subseteq L^2(S)$ such that, for each $G = \{g_a: |a| \leq p\} \subseteq \Gamma'$,
$$\langle L_n, G \rangle = \sum_{|a| \leq p} \int f_{a,n}(x) g_a(x) \, dx \quad \text{with} \quad \|L_n\| = \sum_{|a| \leq p} \|f_{a,n}\|_2.$$  

In particular, for $\varphi \in K\{M_p\}$,
$$\langle L_n, \theta(\varphi) \rangle = \langle T_n, \varphi \rangle = \sum_{|a| \leq p} \int f_{a,n}(x) M_p(x) D^a \varphi(x) \, dx$$
or
$$T_n = \sum_{|a| \leq p} (-1)^{|a|} D^a (M_p f_{a,n}).$$

Since
$$\|L_n\| = \sum_{|a| \leq p} \|f_{a,n}\|_2 \to 0,$$
(ii) is established.

To show that (ii) implies (i) note that, for $\varphi \in K\{M_p\}$,
$$|\langle T_n, \varphi \rangle| \leq \sum_{|a| \leq p} \int |f_{a,n}(x)| M_p(x) |D^a \varphi(x)| \, dx \leq \|\varphi\|_{p,2} \sum_{|a| \leq p} \|f_{a,n}\|_2$$
so that, by (ii), $T_n \to 0$ weakly in $K\{M_p\}'$.

Remark. The proof of Theorem 6 presented here differs from the proof of the characterization of convergent sequences in $\mathcal{D}'_K$ given in [5] (Theorem XXIII of Chapter III) or in [2] (Theorem 19, Section 5 of Chapter 3). The proofs in [5] and [2] seem to rely on using $L^2$-space methods to extend the linear functional $L_n$, and then obtain the fact that $L_n \to 0$ weakly in $\Gamma'$. By employing the result in 1.6.4 of [3], we obtain immediately that actually $L_n \to 0$ strongly in $\Gamma'$. After making this observation, we see that it is really not important to use the $L^2$-space. That is, we could use the norms $\{\|\|_{p,1}\}$ (or $\{\|\|_p\}$) and let $\Gamma'$ be a direct sum of $L^1(S)$ (or $C(S)$) and obtain a representation as in (ii) with $f_{a,n} \in L^\infty(S)$ and $\lim \|f_{a,n}\|_\infty = 0$ (or $f_{a,n}$ bounded measures with $\text{var}(f_{a,n}) \to 0$).

We now give a characterization of sequential convergence in $K\{M_p\}'$ in terms of regularizations. For this result we impose an additional condition on the sequence $\{M_p\}$. The sequence $\{M_p\}$ satisfies the following condition:
(F) Each $M_p$ is finite valued, $M_p(x) = M_p(-x)$ for $x \in \mathbb{R}^m$, and, for each $p$, there are $p' > p$ and $C_p$ such that

$$M_p(x + h) \leq C_p M_{p'}(x) M_{p'}(h) \quad \text{for } x, h \in \mathbb{R}^m.$$ 

Problems concerning convolution in $K\{M_p\}$-spaces have been treated in [7]. In particular, if condition (F) is satisfied, translation is continuous on $K\{M_p\}$ and regularizations can be formed (see Lemma 1 of [7] and III.3.1 of [3]). Recall that if $T \in \mathcal{D}'$ and $\varphi \in \mathcal{D}$, the regularization of $T$ by $\varphi$ is the function $T \ast \varphi: x \mapsto \langle T, (\tau_x \varphi) \rangle$, where $\tau_x \varphi: y \mapsto \varphi(y - x)$ and $\varphi: x \mapsto \varphi(-x)$. Thus, if $T \in K\{M_p\}'$ and $\varphi \in \mathcal{D}$, then $T \ast \varphi$ is an infinitely differentiable function (see [5], Chapter VI, Theorem XI).

Before stating our result, we introduce some auxiliary spaces. For each positive integer $p$, let $B_p$ be the vector space of all continuous complex-valued functions $f$ on $\mathbb{R}^m$ such that

$$|f|_p = \sup \{|f(x)|/M_p(x): x \in \mathbb{R}^m\} < \infty.$$ 

We equip $B_p$ with the norm $| \cdot |_p$, and note that $B_p$ is a $B$-space under this norm.

**Theorem 7.** Let $\{M_p\}$ satisfy conditions (M), (N) and (F). The following conditions are equivalent:

(i) $T_n \to 0$ in $K\{M_p\}'$ (weakly or strongly);

(ii) condition (ii) of Theorem 6;

(iii) there is a positive integer $q$ such that $T_n \ast \varphi \to 0$ in $B_q$ for each $\varphi \in \mathcal{D}$;

(iv) there exist positive integers $q$ and $l$ and, for each multi-index $a$ with $|a| \leq l$, there exists a sequence $\{f_{a,n}\} \subseteq B_q$ such that

$$\lim_{n} f_{a,n} = 0 \text{ in } B_q \quad \text{and} \quad T_n = \sum_{|a| \leq l} D^a f_{a,n}.$$ 

**Proof.** Since each $M_p$ is finite valued from condition (F), condition (N) implies condition (F) and Theorem 6 gives the equivalence of (i) and (ii).

Suppose (ii) holds. For $\varphi \in \mathcal{D}$, we have

$$|T_n \ast \varphi(x)| = \left| \sum_{|a| \leq l} (-1)^{|a|} \int M_p(y)f_{a,n}(y)D^a\varphi(x-y)dy \right|$$

$$\leq C_q M_q(x) \sum_{|a| \leq l} \int M_q(t)|f_{a,n}(x-t)||D^a\varphi(t)|dt$$

$$\leq C_q M_q(x) ||\varphi||_{q,2} \sum_{|a| \leq l} ||f_{a,n}||_2,$$

where $q = p'$ is given by condition (F), and the Cauchy-Schwartz inequality has been used. By (ii) and (7), $T_n \ast \varphi \to 0$ in $B_q$, and (iii) is established.
Suppose (iii) holds. To establish (iv) we apply Theorem 3 of [1] with the space $B$ in this theorem equal to $B_q$ as above. By the conclusion of this theorem, there is a positive integer $l$ and sequences $\{f_n\}$ and $\{g_n\}$ from $B_q$ such that $\lim f_n = \lim g_n = 0$ in $B_q$ and

$$\langle T_n, \varphi \rangle = \int f_n(x)(1 - \Delta(x))\varphi(x)\,dx + \int g_n(x)\varphi(x)\,dx \quad \text{for } \varphi \in \mathcal{D}. \tag{8}$$

Since $f_n$ and $g_n$ belong to $B_q$, the map

$$\varphi \mapsto \int f_n(x)(1 - \Delta(x))\varphi(x)\,dx + \int g_n(x)\varphi(x)\,dx$$

defines a continuous linear functional on $K\{M_p\}$ (see [3], II.4.2), and equation (8) shows this continuous linear functional agrees with $T_n$ on the dense set $\mathcal{D}$ (see [3], II.2.5). Therefore, equation (8) is valid for $\varphi \in K\{M_p\}$ and (iv) follows.

Suppose (iv) holds. We may assume that $q \geq l$ in (iv) since the injections $B_j \to B_{j+1}$ are continuous. For $\varphi \in K\{M_p\}$,

$$|\langle T_n, \varphi \rangle| \leq \sum_{|a| \leq l} \int |f_{a,n}(x)|D^a\varphi(x)\,dx$$

$$\leq \sup \{|f_{a,n}(x)/M_q(x)| : x \in \mathbb{R}^m, \ |a| \leq l\} \|\varphi\|_{q,1},$$

so that $T_n \to 0$ weakly. That is, (i) holds and the result is established.

Remarks. Note that the spaces in Examples 1-4 satisfy condition (F) so that Theorem 7 is applicable to these spaces.

For $K\{M_p\} = \mathcal{S}$, the equivalence of (i) and (iv) is recorded in Theorem 56 in Chapter 3 of [2]. (See also the remark following Theorem VI of Chapter VII of [5].) No analogues of the regularization condition (iii) seems to be recorded, even for $\mathcal{S}$.

For $K\{M_p\} = \mathcal{H}_r$ as in Example 4, the equivalence of (i) and (iv) is given in Theorem 3 of [8].

We conclude by mentioning that it might be possible to alter condition (iii) of Theorem 7 somewhat. Note that $B_q \subseteq B_{q+1}$ with the injection continuous. Thus, if we set

$$B = \bigcup_{q \geq 1} B_q,$$

$B$ may be supplied with the inductive limit topology from the $\{B_q\}$. If this inductive limit is regular (i.e., a set $A \subseteq B$ is bounded iff $A$ is contained in some $B_q$ and bounded in $B_q$), $B$ will be sequentially complete, and, by the proof of (iii) implies (iv) above, we can replace condition (iii) with the condition

(iii') $T_n * \varphi \to 0$ in $B$ for each $\varphi \in \mathcal{D}$.

(See the statement of Theorem 3 in [1].) One possible way of showing that the inductive limit is regular would be to show that, for each $q$, there is a $q' > q$ such that the injection $B_q \to B_{q'}$ is compact (see [6]); however, we have not been able to establish this.
REFERENCES


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