REMARKS ON INFINITE PRODUCTS OF FINITELY ADDITIVE MEASURES

BY

LILIANA WAJDA (WROCLAW)

The present paper contains some remarks about infinitely direct (i.e., satisfying formula (2)) products of measures. By a measure we mean here a normed (finitely) additive non-negative set function on a field of sets.

Theorem 1 of this paper has been proved by Łomnicki and Ulam (see [3], p. 256). Our proof is based on a different idea and uses the following theorem of Pettis [5] (see also Kisynski [1] and Lipecki [2]):

(i) Every strictly additive (i.e., satisfying formula (3)) non-negative set function on an additive and multiplicative family of sets can be uniquely extended to an additive non-negative set function on the ring generated by this family.

We use the following notation:

\[ \underline{\mu}(Z) = \sup \{ \mu(A) \mid A \subset Z, A \in K \} \]

\[ \overline{\mu}(Z) = \inf \{ \mu(A) \mid A \supset Z, A \in K \} \]

Theorem 2 will be proved with the help of the following theorem formulated by Marczewski and Łoś in [4]:

(ii) If \( K \) is a field of sets, \( \mu \) – a measure on \( K, Z \notin K \), and \([K, Z] \) denotes the field generated by \( K \) and \( Z \), then there exists a measure \( \nu \) on \([K, Z] \) such that \( \nu(Z) = \xi \), where \( \xi \) is an arbitrary value satisfying inequality \( \underline{\mu}(Z) \leq \xi \leq \overline{\mu}(Z) \).

Let \( X_i, K_i, \mu_i (i = 1, 2, \ldots) \) denote an arbitrary set, a field of sub-sets of this set and a measure on \( K_i \), respectively. Let \( \mathcal{X} \) be the family of infinite products \( (A_1 \times A_2 \times \ldots) \), where \( A_i \in K_i \), for each \( i \) and \( K \) is the field of sets generated by \( \mathcal{X} \). By \( K_0 \) we denote the smallest field containing all sets of the form \( (A_1 \times A_2 \times \ldots) \), where \( A_i = X_i \) for almost every \( i = 1, 2, \ldots \). Obviously, \( K_0 \subset K \). It will be also convenient to use the notation

\[ \pi_k: \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{k} X_i, \text{ where } \pi_k(x_1, x_2, \ldots) = (x_1, \ldots, x_k). \]
From the axiom of choice and properties of set theoretical operations we infer the following

**Lemma.** For arbitrary \( A = \bigcup_{i=1}^{m} A^i \) and \( B = \bigcup_{j=1}^{n} B^j \), where \( A^i, B^j \in \mathcal{X} \), there exists \( k_0 \) such that

\[
\pi_k(A \cap B) = \pi_k A \cap \pi_k B \quad \text{for} \ k \geq k_0.
\]

It is well known that, for arbitrary measures \( \mu_i \) on fields \( K_i \), there exists a unique measure \( \mu_0 \) on \( K_0 \) such that for \( (A_1 \times \ldots \times A_k \times X_{k+1} \times \ldots) \in K_0 \), where \( k = 1, 2, \ldots \), we have

\[
(1) \quad \mu_0(A_1 \times \ldots \times A_k \times X_{k+1} \times \ldots) = \mu_1(A_1) \cdots \mu_k(A_k).
\]

A stronger theorem is also true.

**Theorem 1.** For arbitrary (normed finitely additive) measures \( \mu_i \) on fields \( K_i \) there exists the unique measure \( \mu \) on \( K \) such that

\[
(2) \quad \mu(A_1 \times A_2 \times \ldots) = \mu_1(A_1) \mu_2(A_2) \cdots \quad \text{for every} \ A_j \in K_j.
\]

**Proof.** The family of sets

\[
\mathcal{X}_s = \left\{ \bigcup_{i=1}^{m} A^i \mid A^i \in \mathcal{X}, \ m = 1, 2, \ldots \right\}
\]

is additive and multiplicative, and for \( A \in \mathcal{X}_s \) the function

\[
\mu(A) = \lim_{k \to \infty} \mu_0(\pi_k^{-1} \pi_k A)
\]

is well defined on \( \mathcal{X}_s \).

In virtue of lemma, we have

\[
(3) \quad \mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B) \quad \text{for} \ A, B \in \mathcal{X}_s.
\]

By Pettis theorem (i) we extend \( \mu \) to the unique measure on the field generated by \( \mathcal{X} \). It is easy to see that \( \mu \) after extension is the unique measure satisfying (2).

We will use the following notation:

\[
K_0^0 = \left\{ A \mid A = \bigcap_{i=1}^{\infty} X_i, \ \mu_0(A) = \mu_0(A) \right\}.
\]

It is interesting to know whether and under what conditions measure \( \mu_0 \) can be uniquely extended to a measure on \( K \) satisfying (1). The answer to this question is given by

**Theorem 2.** There exist two different measures on \( K \) satisfying (1) if and only if

\[
(4) \quad \lim \sup \{ \sup \{ \mu_n(Z) \mid Z \in K_n, \ Z \neq X_n \} \} = 1.
\]
Proof. Sufficiency. If condition (4) is satisfied, then there exists a sequence \( \{A_i\} \) such that \( X_i = A_i \cap K_i \) and \( \mu_i(A_i) \geq 1 - 1/(k+1)^2 \) for \( i \) non belonging to the sequence \( \{i_k\} \) we put \( A_i = X_i \). Let \( A = \prod_{i=1}^{\infty} A_i \). Then \( \mu_0(A) = 0 \) and \( \overline{\mu}_0(A) \geq 1/2 \), and therefore \( A \in \mathcal{X} \setminus K_0^o \). Hence by (ii) there exist two different measures \( \nu_1 \) and \( \nu_2 \) on \( K_0^o \) which are extensions of \( \mu_0 \).

Necessity. It is easy to see that if (4) is not satisfied, then \( \overline{\mu}_0(A) = 0 \) for every \( A = A_1 \times A_2 \times \ldots \) such that \( A_i \in K_i \) and \( A_i \neq X_i \) for infinitely many \( i \). Therefore \( \mathcal{X} \subseteq K_0^o \) and, consequently, \( K \subseteq K_0^o \). It is obvious that measure \( \mu_0 \) on \( K_0 \) can be uniquely extended to a measure on \( K_0^o \).

Condition (4) is satisfied, e.g., if
(a) measures \( \mu_i \) are two-valued and \( K_i \) are not trivial,
(b) measures \( \mu_i \) vanish on singletons and \( K_i \) are not trivial,
(c) \( K_i = 2^{X_i} \) and \( X_i \) is infinite.

REFERENCES


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