Remarks on algebraic concomitants of the Riemann–Christoffel curvature tensor in a three-dimensional space

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Introduction. In the paper [1] the problem of determination of all regular algebraic concomitants  which are densities in the sense of Weyl, of the Riemann–Christoffel curvature tensor in a three-dimensional space has been solved. The authors apply a method of differential equations.

In the present note we show how one can solve this problem by a tensorial method which does not require any assumptions of regularity of the unknown function.

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Let $R_{ij,kl}$ be a Riemann–Christoffel curvature tensor of type (0,4) in a three-dimensional Riemannian (or pseudo-Riemannian) space $V_3$. It fulfills the identities

\begin{align}
(1) & \quad R_{ij,kl} = R_{kl,ij}, \\
(2) & \quad R_{ij,kl} = -R_{ji,kl},
\end{align}

and the first Bianchi identity

$$R_{[ij,kl]} = 0,$$

which in our case of the three-dimensional space $V_3$, as is well known, is a simple consequence of (1) and (2). So (1) and (2) are the only symmetry properties of $R_{ij,kl}$.

Let $\varepsilon^{ijk}$, $\varepsilon_{ijk}$ be the Ricci symbols in $V_3$ which are tensor densities of weights $-1$ and $1$ respectively. We have

\begin{equation}
(3) \quad \varepsilon^{mjk} \varepsilon_{mpq} = \varepsilon_{pq}^{jk},
\end{equation}

where $\varepsilon_{pq}^{jk}$ is the alternating operator equal to $\delta_p^j \delta_q^k$. Let us put

\begin{equation}
(4) \quad \tilde{K}_{ij} ^{\ d} = R_{kl,pq}^{ijkl} \varepsilon^{kl}_{pq}.
\end{equation}

Conversely in view of (3) and (1), (2) we obtain

\begin{equation}
(5) \quad R_{ij,kl} = \frac{1}{4} \tilde{K}_{ij} ^{\ pq} \varepsilon_{pq}^{ijkl}.
\end{equation}
\( \tilde{R}^{ij} \) is a tensor density of type \((2, 0)\) and of weight \(-2\) submitted to the transformation rule

\[
\tilde{R}^{i'j'} = J^2 A_i^{i'} A_j^{j'} \tilde{R}^{ij}.
\]

By (1) and (4) we have

\[
\tilde{R}^{ij} = \tilde{R}^{ik}.
\]

(5) shows that the correspondence (4) between \(R_{ij,kl}\) and \(\tilde{R}^{ij}\) is \(1 - 1\) and invariant. Thus, these two geometric objects are equivalent in the sense of Wagner or Golab. Consequently, the problem of determination of algebraic concomitants of \(R_{ij,kl}\) is equivalent to that for \(\tilde{R}^{ij}\). For instance, if we are looking for all the concomitants of \(R_{ij,kl}\) which are, densities, we may compute them from \(\tilde{R}^{ij}\) and then, using relations (4), represent them as functions of \(R_{ij,kl}\).

In order to find a general form for the density concomitants \(\sigma\) (of a fixed type) of \(\tilde{R}^{ij}\) let us recall that if \(\sigma(\tilde{R}^{ij}) \neq 0\) is one, then any other density concomitant \(\chi\) (of this type) is of the form

\[
\chi = \varphi(\tilde{R}^{ij}) \sigma,
\]

wherein \(\varphi\) is a scalar concomitant of \(\tilde{R}^{ij}\) (i.e. an absolute invariant of the quantities \(\tilde{R}^{ij}\) \(^{(1)}\)).

By (6), any scalar concomitant of the symmetric tensor density \(\tilde{R}^{ij}\) must fulfil the functional equation

\[
\varphi(J^2 A_i^{i'} A_j^{j'} \tilde{R}^{ij}) = \varphi(\tilde{R}^{ij})
\]

for any matrix \([A_i^{i'}]\) such that \(J = \det(A_i^{i'}) \neq 0\). In matrix notation (8) takes the form

\[
\varphi(J^2 A \tilde{R} A^T) = \varphi(\tilde{R}); \quad A = [A_i^{i'}], \quad \tilde{R} = [\tilde{R}^{ij}],
\]

\(A^T\) – the transposed matrix \(A\).

Putting

\[
B = JA^T; \quad \det B = J^4, \quad A = (\det B)^{-\frac{3}{4}} B^T
\]

we get

\[
\varphi(B^T \tilde{R} B) = \varphi(\tilde{R})
\]

for any matrix \(B\) with a positive determinant \(^{(2)}\). (10) may be treated as an functional equation for the invariants of the quadratic form

\[
\tilde{R}^{ij} \xi_i \xi_j
\]

\(^{(1)}\) Because \(\chi/\sigma\) is a scalar.

\(^{(2)}\) The positiveness of \(\det B\) does not influence the solutions of (10) because \((-B)^T \tilde{R} (-B) = B^T \tilde{R} B\) and if \(\det B < 0\), then \(\det (-B) > 0\).
and the only solutions are functions of the rank $r$ and the signature $s$ of this form, which we refer to the quantities $\tilde{R}^{ij}$.

Thus we have shown that

$$\varphi(\tilde{R}^{ij}) = f(r, s),$$

where $f$ is an arbitrary function. If $\det \tilde{R} \neq 0$, then $r = 3$ and $r$ may be omitted in (12).

Since the tensor $R_{ij, kl}$ has an even number of indices, it cannot admit any other types of density concomitants except the Weyl densities [4]

$$\sigma' = |J|^{-w} \sigma; \quad w \text{ is the weight of } \sigma.$$

For such density concomitants $\sigma = \sigma(\tilde{R}^{ij})$ we get the following matrix functional equation:

$$\sigma(J^2 A \tilde{R} A^T) = |J|^{-w} \sigma(\tilde{R}).$$

As before, this equation is equivalent to the following:

$$\sigma(B^T \tilde{R} B) = |B|^{-w} \sigma(\tilde{R}).$$

It has been shown in [4] that equation of type (14) has no non-vanishing solution if $\det \tilde{R} = 0$ (\textsuperscript{3}).

On the other hand, if $\det \tilde{R} \neq 0$, then by (6) $\sigma = \det(\tilde{R}^{ij})$ is a non-vanishing Weyl density of weight $-4$, i.e.

$$\det(\tilde{R}^{ij}) = |J|^4 \det(\tilde{R}^{ij}).$$

Now we come to the conclusion that every density concomitant $\chi$ of a weight $w$ of $\tilde{R}^{ij}$ is of the form

$$\chi = f(s) |\det(\tilde{R}^{ij})|^{-w/4},$$

where $s$ is the signature of the form (11) and $f$ an arbitrary function. One can compute $\det(\tilde{R}^{ij})$ in terms of $R_{ij, kl}$. We have

$$\det(\tilde{R}^{ij}) = \frac{1}{3!} \tilde{R}^{i_1 j_1} \tilde{R}^{i_2 j_2} \tilde{R}^{i_3 j_3} \epsilon_{i_1 i_2 i_3} \epsilon_{j_1 j_2 j_3},$$

and in consequence, by using (3) and (4), we get

$$\det(\tilde{R}^{ij}) = \frac{3}{2} R_{i_1 j_1, k_1 l_1} R_{i_2 j_2, k_2 l_2} R_{i_3 j_3, k_3 l_3} \epsilon^{i_1 i_2 i_3} \epsilon^{j_1 j_2 j_3} \epsilon^{k_1 k_2 k_3} \epsilon^{l_1 l_2 l_3}.$$

Remark 1. Formula (15) provides the densities which are concomitants of the tensor $R_{ij, kl}$ itself. But we may ask for densities as combined concomitants of the pair

$$\{R_{ij, kl}, g_{ij}\};$$

(\textsuperscript{3}) This is equivalent to the fact that a singular quadratic form (11) has no relative invariant.
$g_{ij}$ being the metric tensor in $V_3$. Since each $V_n$ is (locally) conformally an Euclidean one, we have

$$R_{ij, kl} = -\frac{1}{n-2} (R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) - \frac{K}{(n-1)(n-2)} (g_{jk}g_{il} - g_{jl}g_{ik}) ,$$

where $R_{ij}$ is the Ricci tensor and $K$ the scalar curvature, and consequently the pair (16) is equivalent to the pair

$$(17) \quad \{R_{ij}, g_{ij}\} .$$

(17) consists of two covariant symmetric tensors of type $(0, 2)$ and the algebraic concomitants (including densities) of such a pair have been found in [4].

Remark 2. The quantities $\tilde{R}^{ij}$ and $R_{ij}$ are connected with each other by the formula

$$R_{ij} = -\frac{i}{2} \hat{e}_{ikl} \hat{e}_{ipq} \bar{g}^{kp} \tilde{R}^{lj} .$$

Remark 3. By replacing each pair $(i, j)$ of skew-symmetric indices by one single index $a$ such that $(a, i, j)$ is an even permutation of $(1, 2, 3)$ we get from the essential components of $R_{ij, kl}$ a $3 \times 3$ matrix $[R_{ab}]$ by putting $R_{ab} = R_{ij, kl}$ (see [2], [3]). From (4) it follows that

$$[\tilde{R}^{ij}] = 4[R_{ab}] ,$$

which may be useful for computing $\det(\tilde{R}^{ij})$.

Remark 4. The factor $f(s)$ in (15) does not occur in the result obtained in [1], because $s$ is not a differentiable function of components $R_{ij, kl}$.

Remark 5. The above method of determining algebraic concomitants of density type for $R_{ij, kl}$ in a three-dimensional space $V_3$ may be applied, after some modifications, to the problem of determining such concomitants of a tensor

$$R_{i_1 \ldots i_n, j_1 \ldots j_{n-1}}^{i_1, \ldots, j_1 \ldots j_{n-1} = 1, \ldots, n}$$

for $n$ odd, arbitrary and $\geq 3$ (4), which has the following symmetry properties

$$R_{i_1 \ldots i_{n-1}, j_1 \ldots j_{n-1}} = R_{i_1 \ldots i_{n-1}, j_1 \ldots j_{n-1}} ,$$

$$R_{i_1 \ldots i_{n-1}, j_1 \ldots j_{n-1}} = 0 .$$

(4) In the case $n = 3$ no modifications are needed.
References


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