On the existence of a convex solution of the functional equation \( \varphi(x) = h(x, \varphi[f(x)]) \)

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Abstract. In this paper we consider the functional equation \( \varphi(x) = h(x, \varphi[f(x)]) \). Under some conditions on given functions \( f \) and \( h \) we obtain the existence of a convex solution \( \varphi: (-a, a) \to \mathbb{R} \) such that \( \varphi(0) = 0 \). It is assumed that \( f(0) = 0 \).

In the present paper we consider the problem of the existence of a convex solution of the functional equation

\[
\varphi(x) = h(x, \varphi[f(x)]),
\]

where \( f \) and \( h \) are given and \( \varphi \) is an unknown function.

A real function \( \psi \) defined in a convex set \( D \subseteq \mathbb{R}^n \) is convex iff for all \( x, y \in D \) and \( \lambda \in (0, 1) \)

\[
\psi(\lambda x + (1-\lambda)y) \leq \lambda \psi(x) + (1-\lambda) \psi(y).
\]

We assume that

(i) \( f \) is increasing, convex in an interval \( I = (-a, a) \) and

\[
f(0) = 0, \quad f(x) < x \quad \text{for } 0 < x < a,
\]

(ii) \( \Omega \subseteq \mathbb{R}^2 \) is a convex set such that \( (0, 0) \in \Omega \); \( h \) is increasing with respect to each variable and convex in \( \Omega \), and \( h(0, 0) = 0 \),

(iii) for every \( x \in I \), \( h(f(x), \Omega_{f(x)}) \subseteq \Omega_x \), where \( \Omega_x = \{y : (x, y) \in \Omega\} \).

Remark 1. The convexity of \( \Omega \) implies that the function \( \alpha(x) = \inf \Omega_x \) is convex in \( I \) and \( \beta(x) = \sup \Omega_x \) is concave in \( I \). Moreover, if for a certain \( x_0 \in I \) we have \( \alpha(x_0) = -\infty \), then \( \alpha(x) = -\infty \) for every \( x \in I \). Similarly, if for a \( x_0 \in I \) we have \( \beta(x_0) = +\infty \), then \( \beta = +\infty \) in \( I \).

Thus we may confine our considerations to the following two cases: \( \beta < +\infty \) and \( \beta = +\infty \).

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(1) Here \( \mathbb{R}^n \) is a linear metric space with the operations and the metric \( \rho \) defined as follows. Let \( x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), and let \( \lambda \in \mathbb{R} \). Then \( x + y = (x_1 + y_1, \ldots, x_n + y_n), \lambda x = (\lambda x_1, \ldots, \lambda x_n) \) and \( \rho(x, y) = [(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2]^{1/2} \).
1. In this section we consider the simpler case: $\beta < + \infty$. We shall prove the following

**Theorem 1.** Suppose that $\Omega$ is closed and let conditions (i)-(iii) be fulfilled. If for a certain $x_0 \in I$ we have $\sup \Omega_{x_0} < + \infty$, then there exists at least one increasing and convex function $\varphi: I \to \mathbb{R}$ such that $\varphi(0) = 0$, fulfilling equation (1) in $I$.

**Proof.** Suppose that there exists a positive number $c \leq a$ such that

$$a(x) = \inf \Omega_x \leq 0, \quad x \in (0, c),$$

and let us put

$$q_0(x) = 0, \quad x \in (0, c).$$

Next, we define the sequence $q_n$ by the recurrent relation

$$q_n(x) = h(x, q_{n-1}[f(x)]), \quad n = 1, 2, \ldots$$

It follows from (ii) and (iii) that $\beta(x) \geq 0$ for $x \in I$. Thus we have $a(x) \leq q_n(x) \leq \beta(x)$ for $x \in (0, c)$. This together with $f(x) < x$ yields $q_n[\varphi(x)] \in \Omega_x$ for $x \in (0, c)$. Suppose that for a certain $n \geq 1$ and for all $x \in (0, c)$ we have $q_{n-1}[f(x)] \in \Omega_x$. In view of (4) this means that $q_n$ is well defined in $(0, c)$. Then $q_{n-1}[f^2(x)] \in \Omega_{f(x)}$ and according to (4) and (iii) we get

$$q_n[f(x)] = h[f(x), q_{n-1}[f^2(x)]] \in h[f(x), \Omega_{f(x)}] \subset \Omega_x.$$ 

Hence $q_n[f(x)] \in \Omega_x$ for $x \in (0, c)$. We prove by induction that $q_n[f(x)] \in \Omega_x$ for each $n$, and from (4) it follows that $q_n$ is well defined in $(0, c)$ for each $n$. It follows from (i) and (ii) (induction) that $q_n$ is an increasing sequence of increasing and convex functions in $(0, c)$. Since $\beta < + \infty$ (cf. Remark 1), $q_n(x)$ is bounded for every $x \in (0, c)$. Thus there exists a $\varphi(x) = \lim_{n \to \infty} q_n(x)$ for $x \in (0, c)$ and, evidently, $\varphi$ is increasing and convex in $(0, c)$. Taking into account (3), (4) and (ii), we obtain $\varphi(0) = 0$. Letting $n \to \infty$ in (4), we see that $\varphi$ satisfies equation (1) in $(0, c)$. Using (i), (iii) and equation (1), we can extend this solution onto the whole interval $I$ (compare M. Kuczma (2), the proof of a theorem of Kordylewski). For simplicity we denote this extension by $\varphi$. We shall prove that $\varphi$ is increasing and convex in $I$. Let $u$ be the supremum of all $t$ such that $\varphi$ is increasing in $(0, t)$. For the indirect proof suppose that $u < a$. Since $f(u) < u$, it follows from the continuity of $f$ that there exists a $u_1 > u$ such that $f(x) < u$ for $x \in (0, u_1)$. Thus, in view of (i) and (ii), we have for $0 \leq x_1 < x_2 < u_1$

$$\varphi(x_1) = h(x_1, \varphi[f(x_1)]) \leq h(x_2, \varphi[f(x_1)]) \leq h(x_2, \varphi[f(x_2)]) = \varphi(x_2),$$

i.e., \( \varphi \) is increasing in \( (0, u_1) \). This contradiction completes the proof of the monotonicity of \( \varphi \) in \( I \).

Now we denote by \( u \) the supremum of all \( t \) such that \( \varphi \) is convex in \( (0, t) \) and suppose that \( u < a \). Since \( f(u) < u \), it follows from the continuity of \( f \) that there exists a \( u_1 > u \) such that \( f(x) < u \) for \( x \in (0, u_1) \).

Now from the monotonicity of \( \varphi \) and from conditions (i), (ii) we have for \( 0 \leq x_k < u_1, \lambda_k > 0, \lambda_1 + \lambda_2 = 1, \ k = 1, 2 \)

\[
\varphi(\lambda_1 x_1 + \lambda_2 x_2) = h(\lambda_1 x_1 + \lambda_2 x_2, \varphi[f(\lambda_1 x_1 + \lambda_2 x_2)]) \\
\leq h(\lambda_1 x_1 + \lambda_2 x_2, \varphi[f(x_1) + f(x_2)]) \\
\leq h(\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 \varphi[f(x_1)] + \lambda_2 \varphi[f(x_2)]) \\
\leq \lambda_1 h(x_1, \varphi[f(x_1)]) + \lambda_2 h(x_2, \varphi[f(x_2)]) \\
= \lambda_1 \varphi(x_1) + \lambda_2 \varphi(x_2).
\]

Thus \( \varphi \) is convex in \( (0, u_1) \). This contradiction proves that we must have \( u = a \), or that \( \varphi \) is convex in \( I \).

2° Now, suppose that there is no a \( \epsilon > 0 \) such that (2) holds. Then according to the convexity of \( \Omega \), the function \( a(x) = \inf \Omega_x \) has the following properties (cf. Remark 1):

(5) \[ a(0) = 0, \quad a \text{ is increasing and convex in } I. \]

We define

(6) \[ \varphi_0(x) = a(x), \quad x \in I. \]

Using (i)-(iii), it is easy to verify (induction) that the sequence (4) with \( \varphi_0 \) defined above is well defined for \( x \in I \) and forms an increasing sequence of increasing and convex functions in \( I \) and such that \( \varphi_n(0) = 0 \). Moreover, \( \varphi_n(x) \leq \beta[f^{-1}(x)] < \infty \) for \( x \in I \). Thus, the function \( \varphi(x) = \lim_{n \to \infty} \varphi_n(x) \) for \( x \in I \) is increasing, convex, fulfills equation (1) in \( I \) and condition \( \varphi(0) = 0 \). This completes the proof.

2. In this section we assume that

(iv) for every \( x \in I \), \( \sup \Omega_x = +\infty \) and there exists a \( \delta > 0 \) such that \( \inf \Omega_x \leq 0 \) for \( x \in (0, \delta) \).

It follows from (ii) and (iv) that there exist partial derivatives:

\[
\frac{h_0'(0+, 0)}{x} = \lim_{x \to 0^+} \frac{h(x, 0)}{x}, \quad \frac{h_0'(0, 0+)}{y} = \lim_{y \to 0^+} \frac{h(0, y)}{y}.
\]

By (i) we have

\[
f'(0+) = \lim_{x \to 0^+} \frac{f(x)}{x}.
\]

We shall prove the following result.
THEOREM 2. Let conditions (i)-(iv) be fulfilled. If

\[ f'(0+)h'_\varphi(0, 0+) < 1, \]

then there exists an increasing and convex function \( \varphi : I \to R \), fulfilling equation (1) in \( I \) and condition \( \varphi(0) = 0 \).

Proof. For an \( \varepsilon > 0 \) we denote

\[ k = h'_z(0+, 0+) + \varepsilon, \quad l = h'_\varphi(0, 0+) + \varepsilon, \quad s = f'(0+) + \varepsilon. \]

In view of (7) we can choose the \( \varepsilon > 0 \) so small that

\[ sl < 1. \]

It follows from (i) and (ii) that there exists a \( b, 0 < b < \delta \), such that

\[ h(x, y) \leq kx + ly, \quad x, y \in (0, b) \]

and

\[ f(x) \leq sx, \quad x \in (0, b). \]

Let us put

\[ m = k(1-sl)^{-1}, \]

\[ c = \min(b, bm^{-1}) \]

and denote by \( D \) the set

\[ D = \{(x, y) : 0 \leq x \leq c, \ 0 \leq y \leq mx\}. \]

It follows from (12) that \( D \subset \Omega \). Let \( D_x = \{y: (x, y) \in D\} \). Evidently, \( D_x = (0, mx) \). We shall show that

\[ h(f(x), D_{f(x)}) \subset D_x, \quad x \in (0, c). \]

Take \( y \in D_{f(x)} = (0, mf(x)) \). Then by (ii), (9), (i), (10) and (11) we obtain

\[ 0 \leq h(f(x), y) \leq kf(x) + ly \leq kx + lmf(x) \leq (k + slm)x = mx \]

and (13) has been proved. Evidently, \( D \) is closed and convex. If we put \( \Omega = D \), then all the assumptions of Theorem 1 will be fulfilled. Thus there exists an increasing and convex function \( \varphi : (0, c) \to R \), fulfilling equation (1) in \( (0, c) \) and condition \( \varphi(0) = 0 \). This solution has a unique extension onto the whole interval \( I \), which may easily be obtained by using (iii) and equation (1) (compare M. Kuczma (4)). A similar argument as in Theorem 1 proves that this extension is increasing and convex in \( I \). This completes the proof.

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