ASSOCIATE AND PSEUDOASSOCIATE SETS
IN LCA GROUPS, II

BY

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1. Introduction. Let $G$ be a compact Abelian group with dual group $\Gamma$. A compact subset $K$ of $G$ and a subset $\Lambda$ of $\Gamma$ are associated (respectively, pseudoassociated) if for each $\varphi \in B(\Lambda)$ (respectively, $\varphi \in l^\infty(\Lambda)$) there exists a measure $\mu$ (respectively, a pseudomeasure $S$) concentrated on $K$ and such that $\mu|_{\Lambda} = \varphi$ (respectively, $S|_{\Lambda} = \varphi$). Professor S. Hartman has asked whether there is a relationship between the notions of associate and pseudoassociate. In a previous note [1] we proved the existence of pairs $K, \Lambda$ that were pseudoassociate, but not associate. In this note we construct a pair $K, \Lambda$ that is associate, but not pseudoassociate.

We now describe the construction. We work on the circle group $T$. Let $K$ be a Kronecker set, and let $\Lambda$ be a union of arithmetic progressions of increasing length. In the next section we show that $K$ and $\Lambda$ may be constructed in such a way that $K$ and $\Lambda$ are associate. Let us now point out why such $K$ and $\Lambda$ can never be pseudoassociate.

Since $\Lambda$ has arbitrarily long arithmetic progressions, $B(\Lambda) \neq l^\infty(\Lambda)$, as is well known. Since $K$ is a Kronecker set, $M(K) = PM(K)$ by a result of Varopoulos (see [2]). Hence

$$PM(K)^{\wedge}|_{\Lambda} = M(K)^{\wedge}|_{\Lambda} \subseteq B(\Lambda) \neq l^\infty(\Lambda).$$

A necessary and sufficient condition for $K$ and $\Lambda$ to be associate is that there exists a number $\delta > 0$ such that, for all trigonometric polynomials $f$ with frequencies in $\Lambda$,

$$\delta \|f\|_{\infty} \leq \sup \{|f(x)| : x \in K\},$$

but our construction can be simplified only slightly using this fact.

2. The construction. Let $\lambda(1) = 0$, and let $x \in T$ be any element of infinite order. Let $E(1)$ be any finite set of integers such that, for each $z \in T$, there exists $\gamma \in E(1)$ with $|z - \langle \gamma, x \rangle| < 2^{-11}$. Let $\delta(1) = 2^{-11}$. Then for each probability measure $\nu$ on $T$ there exists a probability meas-
ure \sigma on \mathbb{K}(1) = \{x\} \text{ such that } \hat{\nu} = \hat{\sigma} \text{ on } \lambda(1). \text{ Let } U(1) \text{ be a compact neighborhood of } 0 \in T \text{ such that } |1 - \langle \gamma, u \rangle| < 2^{-15} \text{ for } u \in U(1) \text{ and } \gamma \in \mathcal{E}(1). \text{ This begins our inductive construction.}

Suppose that \( n \geq 1 \), and that non-negative integers \( \lambda(1), \lambda(2), \ldots, \lambda(n) \), finite subsets \( \mathcal{E}(1) \subseteq \mathcal{E}(2) \subseteq \ldots \subseteq \mathcal{E}(n) \) of integers, independent finite subsets \( \mathbb{K}(1) \subseteq \mathbb{K}(2) \subseteq \ldots \subseteq \mathbb{K}(n) \) of \( T \), compact neighborhoods \( U(1), U(2), \ldots, U(n) \) of \( 0 \in T \) and numbers \( \delta(1) > \delta(2) > \ldots > \delta(n) > 0 \) have been found so that the following conditions hold:

(1) \[ j \lambda(j) < \lambda(j+1) \quad \text{for } 1 \leq j < n; \]

(2) \[ \lambda(j+1) U(j) = T \quad \text{for } 1 \leq j < n; \]

(3) \[ \delta(j) + |1 - \langle k \lambda(j+1), x \rangle| < 2^{-j-9} \quad \text{for } x \in K(j) \text{ and } 1 \leq k \leq j < n; \]

(4) \[ \text{for } 1 \leq j \leq n \text{ and each probability measure } \nu \text{ on } T, \text{ there exists a probability measure } \sigma \text{ on } K(j) \text{ such that} \]

\[ \delta(j) + \sum_{k=1}^{j} |\hat{\nu}(k \lambda(k)) - \hat{\sigma}(k \lambda(k))| < 2^{-k-8} \quad \text{for } 1 \leq k \leq j; \]

(5) \[ |1 - \langle k \lambda(j+1), x \rangle| < \delta(j)/2 \quad \text{for } x \in K(j) \text{ and } 1 \leq k \leq j < n; \]

(6) \[ \text{for } x, y \in K(j), x + U(j) \cap y + U(j) \neq \emptyset \text{ implies } x = y \text{ for } 1 \leq j \leq n; \]

(7) \[ K(j) + U(j) \supseteq K(j+1) \quad \text{for } 1 \leq j < n; \]

(8) \[ \text{Int } U(j) \supseteq U(j+1) + U(j+2) + \ldots + U(l) \quad \text{for } 1 \leq j < l \leq n; \]

(9) \[ \text{for each } f: K(j) \rightarrow T, \text{ there exists } \gamma \in \mathcal{E}(j) \text{ such that} \]

\[ \delta(j) + |f(x) - \langle \gamma, x \rangle| < 2^{-j-9} \quad \text{for } x \in K(j) \text{ and } 1 \leq j \leq n; \]

(10) \[ |1 - \langle x, \gamma \rangle| < \delta(j)/3j \quad \text{for } x \in U(j), \gamma \in \mathcal{E}(j) \text{ and } 1 \leq j \leq n; \]

(11) \[ |1 - \langle x, j \lambda(k) \rangle| < \delta(k)/3k \quad \text{for } x \in U(k) \text{ and } 1 \leq j \leq k \leq n. \]

We set \( \Lambda(j) = \{k \lambda(j): 1 \leq k \leq j\} \text{ for } 1 \leq j \leq n. \)

We now produce \( \lambda(n+1) \) and \( K(n+1). \) Let \( N \gg n \lambda(n) \) be so large that \( N U(n) = T. \) Let \( \lambda = \lambda(n+1) \gg N \) be such that

(12) \[ |1 - \langle k \lambda, x \rangle| < \delta(n)/3n \quad \text{for } x \in K(n) \text{ and } 1 \leq k \leq n+1. \]

Let \( \Lambda(n+1) = \{k \lambda: 1 \leq k \leq n+1\}. \) Then, for each probability measure \( \nu \) on \( T, \) there exists a probability measure \( \sigma_\nu \) on \( U(n) \) such that

(13) \[ \sum_{\gamma \in \Lambda(n+1)} |\hat{\nu}(\gamma) - \hat{\sigma}_\nu(\gamma)| < 2^{-n-7}. \]
Let $F = K(n) + U(n)$. A straightforward argument shows that there exists a finite set $K \subseteq F$ such that for each probability measure $\sigma_k$ on $F$ there is a probability measure $\sigma_k'$ on $K$ such that

$$\sum_{y \in A(k)} |\hat{\sigma}_k(y) - \hat{\sigma}_k'(y)| < \min\{2^{-k^{-\gamma}}, \delta(n)/3\} \quad \text{for } 1 \leq k \leq n + 1,$$

and such that $K(n+1) = K(n) \cup K$ is independent.

Now let $\nu$ be any probability measure on $T$. Then there exists a probability measure $\sigma_1$ on $K(n)$ such that (4) holds for $1 \leq k \leq n$, and there exists a probability measure $\sigma_2$ on $U(n)$ such that (13) holds. Then (4) and (11) imply

$$2\delta(n)/3 + \sum_{y \in A(k)} |\hat{\nu}(y) - \hat{\sigma}_1(y)\hat{\sigma}_2(y)| < 2^{-k^{-9}} \quad \text{for } 1 \leq k \leq n$$

while (12) and (13) imply

$$2\delta(n)/3 + \sum_{y \in A(k)} |\hat{\nu}(y) - \hat{\sigma}_1(y)\hat{\sigma}_2(y)| < 2^{-n^{-10}}.$$

By our choice of $K$ and (15) there exists a probability measure $\sigma_4$ on $K$ such that

$$\delta(n)/3 + \sum_{y \in A(k)} |\hat{\nu}(y) - \hat{\sigma}_4(y)| < 2^{-k^{-9}} \quad \text{for } 1 \leq k \leq n + 1.$$

This proves that $K(n+1)$, $\lambda(n+1)$ and $\Lambda(n+1)$ have properties (1)-(5) if $0 < \delta(n+1) \leq \delta(n)/3$.

The construction of $E(n+1)$ is straightforward, as is the choice of $U(n+1)$.

The induction is complete.

We let

$$K = \bigcap_{n=1}^{\infty} (K(n) + U(n)).$$

That $K$ is a Kronecker set follows from (7)-(10), exactly as in [3], p. 101 and 102. Let

$$\Lambda = \bigcup_{k=1}^{\infty} \Lambda(k).$$

We now show that $M(K)^\Lambda |_\Lambda = B(\Lambda)$. Let $\varphi \in B(\Lambda)$. Then there exist non-negative measures $\mu_1, \mu_2, \ldots, \mu_4$ such that, on $\Lambda$,

$$\varphi = \hat{\mu}_1 - \hat{\mu}_2 + i\hat{\mu}_3 - i\hat{\mu}_4 \quad \text{and} \quad 2\|\varphi\| \geq \sum_{j=1}^{4} \|\mu_j\|. $$
Let $\sigma_1^{(n)}$, $\sigma_2^{(n)}$, ..., $\sigma_4^{(n)}$ be probability measures on $K(n)$ such that (4) holds for $\nu_j = \|\mu_j\|^{-1}\mu_j$. Let

$$A_n = \bigcup_{j=1}^n A(j),$$

and

$$\omega_n = \|\mu_1\|\sigma_1^{(n)} + \|\mu_2\|\sigma_2^{(n)} + \|\mu_3\|\sigma_3^{(n)} - \|\mu_4\|\sigma_4^{(n)}.$$

Then (4) implies

$$\|\varphi|_{A_n} - \hat{\omega}_n|_{A_n}\|_{B(A_n)} \leq 4 \sum_{k=1}^n 2^{-k-\delta} \|\varphi\| = 2^{-3}\|\varphi\|_{B(A_n)}.$$

Taking weak-* limits, we see that, for each $\varphi \in B(A)$, there exists $\sigma \in M(K)$ such that

(18) \[\|\sigma\| \leq 2\|\varphi\|_{B(A)} \quad \text{and} \quad \|\hat{\sigma}|_A - \varphi\|_{B(A)} < 2^{-3}\|\varphi\|_{B(A)}\].

A standard iteration using (18) shows that $M(K)^*|_A = B(A)$.

The construction is complete.

REFERENCES


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