A. KRZYwicki and A. RYBarski (Wroclaw)

ON AN INTEGRAL INEQUALITY CONNECTED WITH HARDY'S INEQUALITY

Let $F$ denote the class of functions $f = f(t)$, defined and absolutely continuous on the interval $[-1, 1]$, satisfying the conditions $f(-1) = f(1) = 0$ and such that

$$
\int_{-1}^{1} p^a f^2 \, dt < \infty
$$

(1)

where $p = (1 - t)^{-1/2}, \dot{f} = df/dt$ and $a$ is a positive number. We shall prove the following

Theorem If $f \in F$ and

$$
\int_{-1}^{1} p^{a+4} f \, f_0 \, dt = 0,
$$

(2)

where $f_0 = p^{-1-a}$, then the following inequality holds true

$$
\lambda_a \int_{-1}^{1} p^{a+4} f^2 \, dt \leq \int_{-1}^{1} p^a f^2 \, dt
$$

(3)

with $\lambda_a = (1 + a/2)^2$ for $0 < a < 4$ and $\lambda_a = 3a - 3$ for $a \geq 4$. The coefficient $\lambda_a$ in (3) cannot be enlarged. Moreover, for $0 < a < 4$, there is equality in (3) for $f_0 = 0$ only.

If condition (2) is rejected, then we have

$$
(1 + a) \int_{-1}^{1} p^{a+4} f^2 \, dt \leq \int_{-1}^{1} p^a f^2 \, dt
$$

(4)

for any $f \in F$ and there is equality in (4) for $f = \text{const} \, f_0$ only (see [2]). The reason for considering the additional restriction (2) is motivated by some problems of approximation in the theory of differential equations.
Proof of the theorem. We introduce a new independent variable $x$ and a new function $u = u(x)$ which are connected with $t$ and $f$ by the relations

$$ t = \text{th} x/2, \quad f = (\text{ch} x/2)^{-a} u, \quad a = 1 + a/2. $$

As a result of simple calculations we get the identities

$$ J_1 = \int_{-1}^{1} p^{a+4} f^2 \, dt = 1/2 \int_{-\infty}^{\infty} u^2 \, dx = 1/2 \|u\|^2 $$

and

$$ J_2 = \int_{-1}^{1} p^a f^2 \, dt = 2 \|u'\|^2 + a^2 \|u\|^2 + a(1 - a)/2 \int_{-\infty}^{\infty} (\text{ch} x/2)^{-a} u^2 \, dx, $$

where $'$ denotes differentiation with respect to the variable $x$ and $\| \|$ is the standard $L_2(-\infty, \infty)$-norm. When deriving (7) the relation $(\text{th} x/2) u_x \to 0$ as $x \to \pm \infty$ is to be used which, on the other hand, is equivalent to the relation $p^{a+2} f^2 \to 0$ as $t \to \pm 1$, established in [2]. The function $u$ is absolutely continuous on any finite interval of the real line and, due to (7), the norms $\|u\|$ and $\|u'\|$ are finite. The class of all such functions will be called $U$. Let $u \in C_2(-\infty, \infty)$, i.e. $u$ is twice continuously differentiable and vanishing outside of a segment of the $x$-axis. For such $u$ the integral $J_2$ may be expressed in the form

$$ J_2 = 2 (Lu, u) + a^2/2 \|u\|^2, $$

where

$$ Lu = -u'' + a(1 - a)/4 (\text{ch} x/2)^{-2} u $$

and $(\, , \,)$ denotes, as usually, the inner product in $L_2$. The second order differential operator $L$, if considered on $C_2$, is symmetric and it may be extended to a selfadjoint operator $L_0$ with a domain $D(L_0) \subset U$. Its spectrum has been studied by Titchmarsh [3]: it consists of a continuous part which is identical with the positive semiaxis $\kappa \geq 0$ and of a point spectrum whose points $\kappa_r$ lying on the negative semiaxis $\kappa \leq 0$ are given by

$$ \kappa_r = - (a/2 - 1/2 - r)^2, \quad r = 0, 1, \ldots, \lfloor a/2 - 1/2 \rfloor $$

and $[\beta]$ denotes the integer part of $\beta$.

Consider the smallest eigenvalue $\kappa_0 = -a^2/16$ and corresponding eigenfunction $\tilde{u}$. They satisfy the relation

$$ (L_0 \tilde{u}, \tilde{u}) = \kappa_0 \|\tilde{u}\|^2 $$
which is equivalent to \( J_2 = (2 \chi_0 + a^2/2) \| \tilde{u} \|^2 = (1 + a) J_1 \). The last equality is identical with (4), where the \( \le \) sign has been replaced by the \( = \) sign and where \( f = \tilde{f} = (c x/2)^{-a} \). However, the only function of class \( F \) for which there is equality in (4) is the function \( \text{fconst } f_0 \), therefore, due to relation \( \tilde{u} \in U \), we have \( f = \text{const } f_0 \), and \( \tilde{u} = \text{const } u_0 \), where \( u_0 = (c x/2)^{-a} \), is the only eigenfunction of \( L_0 \), corresponding to the eigenvalue \( \chi_0 \).

Let us now remark that it suffices to prove our theorem for functions \( f \in C_2(-1, 1) \) only and then apply the standard limiting procedure. When working with corresponding functions \( u \in U \), we can restrict ourselves to the class \( C_2(- \infty, \infty) \).

Assume \( u \in C_2(- \infty, \infty) \). The orthogonality condition (2), when expressed in terms of \( x \) and \( u \)'s, takes the form

\[
(u, u_0) = 0. \tag{11}
\]

We make now use of spectral representation of the operator \( L_0 \). Due to (11), we have

\[
(L_0 u, u) \geq \mu \| u \|^2 \tag{12}
\]

for any \( u \in C_2(- \infty, \infty) \) where \( \mu \) is the point of the spectrum of \( L_0 \) which is different from and nearest to \( \chi_0 \), i.e. \( \mu = 0 \) for \( 0 < a < 4 \) and \( \mu = \chi_0 = -1/4(a - 3)^2 \) for \( a \geq 4 \). Due to (8) and the identity \( L_0 u = Lu \) for \( u \in C_2(- \infty, \infty) \), the inequality (12) may be written as \( J_2 \geq \lambda_0 J_1 \), \( \lambda_0 = 4 \mu + a^2 \), \( f \in C_2(-1, 1) \), which is the desired inequality (3).

Now for \( 0 < a < 4 \), \( \mu = 0 \) is a point of the continuous part of the spectrum and there is no function other than zero for which there would be equality in (3) with both integrals convergent. If \( a \geq 4 \), then \( \mu = \chi_1 \) and an eigenfunction corresponding to this eigenvalue turns (3) to equality. This completes the proof of the theorem.

A particular case, \( a = 1 \), will be considered now under the additional assumption that \( f \in F \) is an even function. In this case inequality (3) has the form

\[
9/4 \int_{-1}^{1} p^5 f^2 dt \leq \int_{-1}^{1} pf^2 dt \tag{13}
\]

and the orthogonality condition becomes

\[
\int_{-1}^{1} p^3 f dt = 0. \tag{14}
\]

We shall show that in this particular case our theorem is equivalent to Hardy's theorem [1].
We shall make use of the identity
\[
\int_{-1}^{1} pf^2 dt - 2 \int_{-1}^{1} p^5 f^5 dt = \int_{-1}^{1} p^{-3} \hat{h}^2 dt,
\]
(see [2], formula (11)), where \( h = p^2 f \) with \( f \in F \). Let \( \{T_k\}, k = 0, 1, 2, \ldots \), denote the system of Chebyshev’s polynomials. Denoting by \( a_k \) and \( b_k \), coefficients with respect to \( \{T_k\} \) of functions \( h \) and \( p^{-2} \hat{h} \) resp., i.e.
\[
\begin{align*}
\pi a_0 &= \int_{-1}^{1} phT_0 dt, & \pi a_k &= 2 \int_{-1}^{1} phT_k dt, \\
\pi b_0 &= \int_{-1}^{1} p^{-1} \hat{h}T_0 dt, & \pi b_k &= 2 \int_{-1}^{1} p^{-1} \hat{h}T_k dt,
\end{align*}
\]
(16)
we have, due to Parseval’s relation,
\[
\begin{align*}
I_1 &= 2/\pi \int_{-1}^{1} ph^2 dt = 2a_0^2 + \sum_{k \geq 1} a_k^2, \\
I_2 &= 2/\pi \int_{-1}^{1} p^{-3} \hat{h}^2 dt = 2b_0^2 + \sum_{k \geq 1} b_k^2.
\end{align*}
\]
We have also the following identities
\[
a_1 = 2b_0, \quad 2b_k = (k+1)a_{k+1} - (k-1)a_{k-1}
\]
(18)
for \( k = 1, 2, \ldots \). These identities follow simply when performing integration by parts in (16). Due to relations (18) we can express the coefficients \( a_k \) in terms of \( b_k \).

The orthogonality condition (14) is now equivalent to \( a_0 = 0 \). We have also relations \( a_1 = a_3 = \ldots = 0 \) and \( b_0 = b_2 = \ldots = 0 \) implied by the assumption \( h \) to be an even function. If we now put \( \beta_k = b_{2k-1} \), \( B_k = \beta_1 + \ldots + \beta_k \) we can then rewrite \( I_1 \) and \( I_2 \) in the form
\[
\begin{align*}
I_1 &= \sum_{k \geq 1} (B_k/k)^2, & I_2 &= \sum_{k \geq 1} \beta_k^2.
\end{align*}
\]
(19)
According to Hardy’s theorem we have \( I_2 < 4I_1 \) if only not all \( \beta_k \) are zero; here the coefficient 4 is the best one. The inequality (13) is now an immediate consequence of the identity (15). Conversely, Hardy’s theorem follows simply from inequality (13) and identity (15) expressed in terms of \( \beta_k \).

The above arguments may be modified so as to apply to the general case considered in our theorem, however, a generalization of Hardy’s theorem is then needed. We hope to return to this problem elsewhere.
On an integral inequality

References


MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCŁAW
INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

Received on 22. 12. 1967