TWO SUFFICIENT CONDITIONS FOR THE MACLANE CLASS \( \mathcal{A} \)

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Let \( \mathcal{A} \) be the MacLane class of non-constant functions which are analytic in \( |z| < 1 \) and have asymptotic values in a dense set of points of \( |z|^\prime = 1 \). MacLane [3], p. 46, showed that if

\[
M(r, f) = \sup_{|z| = r} |f(z)| \quad (0 < r < 1)
\]

for a non-constant function \( f(z) \) analytic in \( |z| < 1 \), then

\[
\int_{0}^{1} (1 - r) \log^{+} M(r, f) dr < \infty
\]

is sufficient to guarantee \( f(z) \in \mathcal{A} \). MacLane [3], p. 51, further proved that if

\[
f(z) = \sum_{n=0}^{\infty} a_{n} z^{n}
\]

is such that, for some \( \lambda \) \((0 < \lambda < \frac{1}{3})\),

\[
\log^{+} |a_{n}| < n^{\lambda} \quad (n > n_{0}),
\]

then \( f(z) \in \mathcal{A} \). Thus, if order \( \varrho \) of \( f(z) \), defined as

\[
\varrho = \limsup_{r \to 1, r < 1} \frac{\log^{+} \log^{+} M(r, f)}{-\log(1 - r)},
\]

satisfies \( 0 < \varrho < 2 \), then \( f(z) \in \mathcal{A} \).

Hornblower [2] weakened condition (1) and showed that if \( f(z) \) is non-constant analytic in \( |z| < 1 \) such that

\[
\int_{0}^{1} \log^{+} \log^{+} M(r, f) dr < \infty,
\]

then \( f(z) \in \mathcal{A} \).
The purpose of the present paper is to weaken (2) and to obtain a sufficient condition on \(|a_n/a_{n-1}|\) such that

\[ f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \quad (a_n \neq 0 \text{ for all } n) \]

is in the class \(\mathcal{A}\).

Our results imply that all non-constant functions, analytic in \(|z| < 1\) and having finite order, are in the class \(\mathcal{A}\). Further, we construct an example to show that there are functions of infinite order which also belong to the class \(\mathcal{A}\).

**Lemma 1.** Let \(f(z)\) be analytic and non-constant in \(|z| < 1\) and let

\[ M(r,f) = \sup_{|z|=r} |f(z)|. \]

If, for some \(a\) \((1 < a < \infty)\),

\[ \log^+ \log^+ M(r,f) = O\left\{(1-r)^{-1} \left(\log \frac{e}{1-r}\right)^{-a}\right\} \quad \text{as } r \to 1, \]

then \(f(z) \in \mathcal{A}\).

**Proof.** It is easily seen that the hypothesis of the lemma implies

\[ \int_0^1 \log^+ \log^+ M(r,f) \, dr < \infty. \]

Therefore \(f(z) \in \mathcal{A}\) now follows from Hornblower's result ([2], Theorem 1).

**Theorem 1.** Let

\[ f(z) = \sum_{n=0}^{\infty} a_n z^{\lambda_n} \quad (|z| < 1) \]

be a non-constant function. If, for some \(\beta\) \((1 < \beta < \infty)\),

\[ \log^+ |a_n| = O \{\lambda_n (\log \lambda_n)^{-1} (\log \log \lambda_n)^{-\beta}\} \quad \text{as } n \to \infty, \]

then \(f(z) \in \mathcal{A}\).

**Proof.** Let us first observe that condition (4) implies that \(f(z)\) is analytic in \(|z| < 1\) and that there exist positive finite constants \(B\) and \(S\) such that, for all \(n > S\),

\[ \log^+ |a_n| < B \lambda_n (\log \lambda_n)^{-1} (\log \log \lambda_n)^{-\beta}. \]

We write

\[ M(r,f) \leq \sum_{n=0}^{\infty} |a_n|r^{\lambda_n} = \sum_{n=0}^{S} |a_n|r^{\lambda_n} + \sum_{n=S+1}^{N} |a_n|r^{\lambda_n} + \sum_{n=N+1}^{\infty} |a_n|r^{\lambda_n}, \]

where
where

\[ N = \left[ \exp \left( \exp \left( \frac{1}{2B} \log \frac{1}{r} \right)^{-1/(\beta+1)} \right) \right]. \]

It follows that

\[ \sum_{n=N+1}^{\infty} |a_n| r^n = o(1) \quad \text{as} \quad r \to 1, \]

for

\[ \sum_{n=N+1}^{\infty} |a_n| r^n < \sum_{n=N+1}^{\infty} \exp \{ B \lambda_n (\log \lambda_n)^{-1} (\log \log \lambda_n)^{-\beta} \} r^n \]

\[ \leq \sum_{n=N+1}^{\infty} \frac{r^{n/2}}{1 - r^{1/2}}, \]

and \( r^{(N+1)/2} (1 - r^{1/2}) \to 0 \) as \( r \to 1 \) in view of the estimate

\[ 1 - r = \left( \log \frac{1}{r} \right) \left( 1 + O \left( \log \frac{1}{r} \right) \right) \]

for the values of \( r \) sufficiently close to 1. Thus, by (5), for all \( r \) satisfying \( r_0 < r < 1 \),

\[ M(r, f) < c(S) + N \max_{n \geq 0} \left\{ \exp \{ B \lambda_n (\log \lambda_n)^{-1} (\log \log \lambda_n)^{-\beta} \} r^n \right\} + o(1), \]

where \( c(S) \) is a constant depending on \( S \). Now, let

\[ g(x, r) = Bx (\log x)^{-1} (\log \log x)^{-\beta} + x \log r. \]

The maximum value of \( g(x, r) \) occurs at the point \( x = x_0 \equiv x_0(r) \) satisfying the equation

\[ B (\log x)^{-1} (\log \log x)^{-\beta} \{ 1 - (\log x)^{-1} - B \beta (\log x)^{-1} (\log \log x)^{-1} \} = \log \frac{1}{r}. \]

It is easily seen that \( x_0(r) \to \infty \) as \( r \to 1 \), so that, for all \( r \) satisfying \( r_1 < r < 1 \) we have

\[ x_0(r) = \exp \left\{ (1 + o(1)) B \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} \right\}. \]

Thus, by (7),

\[ g(x, r) \leq B x_0 (\log x_0)^{-1} (\log \log x_0)^{-\beta} \{ 1 + B \beta (\log \log x_0)^{-1} \}. \]
Using the estimate of $x_0(r)$, this estimate of $g(x, r)$ yields
\[
\log g(x, r) \leq \log x_0 + o(1)
\]
\[
= B(1 + o(1)) \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} + o(1)
\]
for all values of $r$ sufficiently close to 1. Now it follows from (6) that, as $r \to 1$,
\[
\log^+ M(r, f) \leq \log N + \exp \left\{ B(1 + o(1)) \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} + o(1) \right\} + o(1) \leq \exp \left( \frac{1}{2B} \log \frac{1}{r} \right)^{-1} + o(1) \leq \exp \left\{ B(1 + o(1)) \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} + o(1) \right\} + o(1)
\]
\[
= (1 + o(1)) \exp \left\{ B(1 + o(1)) \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} + o(1) \right\}.
\]

Since the right-hand side expression in this inequality is a positive quantity, we have, as $r \to 1$,
\[
\log^+ \log^+ M(r, f) \leq B(1 + o(1)) \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} + o(1)
\]
\[
= O \left\{ \left( \log \frac{1}{r} \right)^{-1} \left( -\log \log \frac{1}{r} \right)^{-\beta} \right\} = O \left\{ (1 - r)^{-1} \left( \log \frac{e}{1-r} \right)^{-\beta} \right\}.
\]

Thus, by Lemma 1, $f(z) \in \mathcal{A}$ and the proof of Theorem 1 is complete.

Remark. Since condition (4) follows from MacLane's condition (2), Theorem 1 provides an improvement to MacLane's result (see [3], p. 51). Further, if
\[
f(z) = \sum_{n=0}^{\infty} a_n z^n
\]
is analytic in $|z| < 1$ and has order $\rho$, then proceeding on the lines of Beuermann [1] or MacLane [3] (p. 47) it is not difficult to prove that
\[
ex = \limsup_{n \to \infty} \frac{\log^+ \log^+ |a_n|}{\log \lambda_n}.
\]

It is clear from (8) that, for functions of finite order, condition (4) is satisfied for any $\beta > 1$. Thus, by Theorem 1, all non-constant functions, analytic in $|z| < 1$ and having finite order, are in $\mathcal{A}$. The same conclusion also follows by Horneblower's result, since a function analytic in $|z| < 1$ and having finite order satisfies (3).
An example of a function in the class $\mathcal{A}$ having infinite order can easily be constructed by help of Theorem 1. Indeed, consider the function

$$g(z) = \sum_{n=0}^{\infty} \exp\left(\lambda_n \left(\log \lambda_n\right)^{-3}\right) z^n,$$

where $\lambda_0 = 0$ and $\{\lambda_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers. Since (4) is satisfied, we infer that $g(z)$ is in the class $\mathcal{A}$, and, by (8), the order of $g(z)$ is infinite.

Let us observe that whereas Theorem 1 provides an example of a function of infinite order in the class $\mathcal{A}$ in terms of a Gap Taylor series, such examples in the closed form can easily be constructed with the use of Hornblower's result. Indeed, the function

$$h(z) = \exp\left(\exp\left(1 - z^{-a}\right)^{-a}\right) \quad (0 < a < 1)$$

is in the class $\mathcal{A}$ in view of (3), and it can easily be seen that the order of $h(z)$ is infinite.

**Lemma 2.** Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a non-constant function analytic in $|z| < 1$ and of order $\varrho$. If

$$\psi(n) = \dfrac{a_n}{a_{n+1}} \geq \frac{1}{e} \quad \text{for all } n > n_0,$$

then

$$1 + \varrho \leq \max(1, \theta),$$

where

$$\theta = \limsup_{n \to \infty} \frac{\log \lambda_n}{\log((\lambda_n - \lambda_{n-1})/\log|a_n/a_{n-1}|)}.$$

**Proof.** The condition $\psi(n) \geq 1/e$ for all $n > n_0$ implies $0 \leq \theta \leq \infty$. Let $\theta < \infty$. For any $\delta$ such that $\theta < \delta < \infty$ we have, for all $n > N = N(\delta)$,

$$\log^+ \left| \dfrac{a_n}{a_{n-1}} \right| < (\lambda_n - \lambda_{n-1}) \lambda_n^{-1/\delta}.$$

Therefore, if $n > \max(N, n_0)$, then

$$\log|a_n| < \log|a_N| + (\lambda_{N+1} - \lambda_N) \lambda_{N+1}^{-1/\delta} + \ldots + (\lambda_n - \lambda_{n-1}) \lambda_n^{-1/\delta}$$

$$= \log|a_N| + \lambda_n^{(\theta-1)/\delta} - \sum_{m=N+1}^{n-1} \lambda_m (\lambda_{m-1}^{1/\delta} - \lambda_m^{-1/\delta}) - \lambda_N \lambda_{N+1}^{-1/\delta}$$

$$= \log|a_N| + \lambda_n^{(\theta-1)/\delta} - \int_{\lambda_{N+1}}^{\lambda_n} n(t) d(t^{-1/\delta}) - \lambda_N \lambda_{N+1}^{-1/\delta},$$

where $n(t) = \lambda_m$ for $\lambda_m < t \leq \lambda_{m+1}$ and $m = N+1, \ldots, n-1.$
Since
\[ \int_{\lambda_{n+1}}^{\lambda_n} n(t) \, dt \, t^{-1/\delta} > - \frac{1}{\delta} \int_{\lambda_{n+1}}^{\lambda_n} t^{-1/\delta} \, dt = - \frac{1}{\delta-1} \{ \lambda_n^{(\delta-1)/\delta} - \lambda_{n+1}^{(\delta-1)/\delta} \}, \]
equation (10), for sufficiently large values of \( n \), gives
\[ \log |a_n| < \log |a_N| + \frac{\delta}{\delta-1} \lambda_n^{(\delta-1)/\delta} - \frac{1}{\delta-1} \lambda_{n+1}^{(\delta-1)/\delta} - \lambda_N \lambda_{n+1}^{(\delta-1)/\delta}. \]

If \( \theta < \delta < 1 \), then (11) and (8) imply \( \epsilon = 0 \), and so (9) obviously holds. Hence, suppose that \( 1 \leq \theta < \delta < \infty \). It follows from (11) that, for sufficiently large values of \( n \),
\[ \log^+ \log^+ |a_n| < \frac{\delta-1}{\delta} \log \lambda_n + o(1), \]
which, in view of (8), implies the inequality
\[ \frac{\epsilon}{1+\epsilon} \leq \frac{\delta-1}{\delta}. \]

Since this inequality holds for every \( \delta > \theta \), we have
\[ \frac{\epsilon}{1+\epsilon} \leq \frac{\theta-1}{\theta}, \]
and so \( 1+\epsilon \leq \theta \). This completes the proof.

Theorem 2. Let
\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]
be a non-constant function. If, for some \( \eta \) \( (0 < \eta < \infty) \),
\[ \log^+ \left| \frac{a_n}{a_{n-1}} \right| = O((\lambda_n - \lambda_{n-1})\lambda_n^{-\eta}) \quad \text{as} \quad n \to \infty, \]
then \( f(z) \in \mathcal{A} \).

Proof. It is easily seen that \( f(z) \) is analytic in \( |z| < 1 \). In view of Lemma 2, condition (12) on \( |a_n|/|a_{n-1}| \) implies that \( f(z) \) is of finite order. The assertion \( f(z) \in \mathcal{A} \) now follows from the remark following Theorem 1.

References


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