SEMILATTICES DO NOT HAVE EQUATIONALLY
COMPACT HULLS

BY

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A semilattice is an algebra with one binary operation \( \wedge \) which is associative, commutative and idempotent. Each semilattice has a natural partial ordering on it, given by \( a \leq b \) iff \( a \wedge b = a \); the symbol \( \vee \) denotes the least upper bound (join) under this partial ordering.

Recall that an algebra \( A \) is equationally compact if every subset \( \Sigma \subseteq A[X]^2 \) (\( A[X] \) is the free extension of \( A \) by the set \( X \) in any equational class containing \( A \)) is contained in the kernel of a homomorphism \( A[X] \rightarrow A \) over \( A \) whenever every finite subset of \( \Sigma \) has this property. An extension \( B \) of an algebra \( A \) is pure if every finite subset of \( A[X]^2 \) is contained in the kernel of a homomorphism \( A[X] \rightarrow B \) over \( A \) whenever it is contained in the kernel of a homomorphism \( A[X] \rightarrow B \) over \( A \). These notions were introduced for general algebras by Mycielski [4] and Węglorz [11], respectively; the present algebraic formulations can be found in Banaschewski and Nelson [1].

The equationally compact semilattices were characterized by Grätzer and Lakser [3] as those semilattices \( S \) which are conditionally complete (i.e. every non-empty subset has a greatest lower bound), in which every chain has a least upper bound, and in which the distributivity law \( a \wedge \bigvee C = \bigvee \{ a \wedge c \mid c \in C \} \) holds for all \( a \in S \) and for all chains \( C \subseteq S \). Bulman-Fleming [2] has proved that every equationally compact semilattice is a retract of a (topologically) compact one, and Taylor [10] has improved this replacing “retract of a compact one” by “retract of a product of finite semilattices”.

It is easy to see that every semilattice has an equationally compact extension; this is provided by the usual embedding of a semilattice \( S \) into \( P(S) \), the power set of \( S \) with the operation of set intersection, which maps \( s \) to \( \{ a \in S \mid a \leq s \} \), and the fact that \( P(S) \) is isomorphic to the \( S \)-th power of the two-element semilattice and hence is equationally compact.
A semilattice $S$ has an equationally compact hull (in the class of all semilattices) iff it has a pure, equationally compact semilattice extension (Banaschewski and Nelson [1], Proposition 2). It is the aim of this note to prove that semilattices do not have equationally compact hulls by exhibiting, for each infinite cardinal $n$, an $n$-element semilattice $S_n$ which has no pure, equationally compact, semilattice extension. (Note that such a semilattice has no pure equationally compact extension at all; see [1], p. 156, Remark 2.)

By Theorem 3.12 of Taylor [8] or Proposition 4 of Banaschewski and Nelson [1], it will then follow that there are pure-irreducible semilattices of arbitrarily high cardinality; in fact, we will see that all $S_n$ are actually pure irreducible.

The reader is referred to [1] for all notions concerning purity and equational compactness which are not defined here.

Definition of $S_n$. For any cardinal number $n$ (finite or infinite), let $S_n$ be the semilattice with underlying set $(n \times \{0, 1\}) \cup \{u, 1\}$ (where $u \notin n \times \{0, 1\}$) and with the operation defined by

$$x \wedge 1 = x \quad \text{for all } x,$$

$$(\lambda, 0) \wedge u = (\lambda, 1) \wedge u = (\lambda, 0) \quad \text{for all } \lambda < n,$$

$$(\lambda, 0) \wedge (\mu, 0) = (\lambda, 0) \wedge (\mu, 1) = (\min(\lambda, \mu), 0) \quad \text{for all } \lambda, \mu < n,$$

$$(\lambda, 1) \wedge (\mu, 1) = \begin{cases} (\min(\lambda, \mu), 0) & \text{if } \lambda \text{ is even and } \mu \text{ is odd or vice versa,} \\ (\min(\lambda, \mu), 1) & \text{otherwise.} \end{cases}$$

The underlying partially ordered set of $S_n$ is depicted in Fig. 1.

First of all, note that, for all $n$, there are no elements $x, y \in S_n$ such that $x \wedge u = y \wedge u = x \wedge y$ and $x \wedge (0, 1) = (0, 1)$ and $y \wedge (1, 1) = (1, 1)$. Indeed, if $x \wedge (0, 1) = (0, 1)$, then either $x = 1$ or $x = (\lambda, 1)$ for some even $\lambda < n$ and, similarly, $y = 1$ or $y = (\mu, 1)$ for some odd $\mu < n$; since $x \wedge u = y \wedge u$, it follows that $x = y = 1$, and thus $x \wedge y = 1 \neq x \wedge u$. Thus

$$\Sigma = \{(x \wedge u, y \wedge u), (x \wedge u, x \wedge y), (x \wedge (0, 1), (0, 1)), (y \wedge (1, 1), (1, 1))\}$$

is a finite subset of $S_n[\{x, y\}]^2$ which is not contained in the kernel of any homomorphism $S_n[\{x, y\}] \to S_n$ over $S_n$.

Now suppose that, for some infinite $n$, $T \supseteq S_n$ is an equationally compact extension. Then it follows from the Grätzer-Lakser result mentioned above that, in $T$, the sets

$$C_0 = \{(\lambda, 1) \mid \lambda \text{ even}\}, \quad C_1 = \{(\lambda, 1) \mid \lambda \text{ odd}\} \quad \text{and} \quad C_2 = n \times \{0\}$$

have least upper bounds. Let $\bar{x} = \vee C_0$, $\bar{y} = \vee C_1$ and $z = \vee C_2$. Then, again by the Grätzer-Lakser result,

$$u \wedge x = \vee \{u \wedge (\lambda, 1) \mid \lambda \text{ even}\} = \vee \{(\lambda, 0) \mid \lambda \text{ even}\} = z.$$
(This uses the fact that \( n \) is infinite.) Similarly, \( u \land \bar{y} = z = \bar{x} \land \bar{y} \). Trivially, \( \bar{x} \land (0, 1) = (0, 1) \) and \( \bar{y} \land (1, 1) = (1, 1) \). Thus (with \( \Sigma \) as above), the homomorphism \( S_n[[x, y]] \to T \) over \( S_n \), mapping \( x \) to \( \bar{x} \) and \( y \) to \( \bar{y} \), contains \( \Sigma \) in its kernel and, consequently, \( T \) is not a pure extension of \( S_n \). It follows that \( S_n \) has no pure, equationally compact extension.

Fig. 1

Now suppose that \( n > 0 \) and that \( \theta \) is a proper congruence on \( S_n \). Then it is easy to see that either \( (u, 1) \in \theta \) or \( ((\lambda, 0), (\lambda + 1, 0)) \in \theta \) for some \( \lambda < n \) or \( ((0, 0), (0, 1)) \in \theta \) or \( ((1, 0), (1, 1)) \in \theta \).

In case \( (u, 1) \in \theta \), let \( \bar{x} = \bar{y} = 1 \).

In case \( ((\lambda, 0), (\lambda + 1, 0)) \in \theta \) for \( \lambda \) even, let \( \bar{x} = (\lambda, 1) \) and \( \bar{y} = (\lambda + 1, 1) \).

In case \( ((\lambda, 0), (\lambda + 1, 0)) \in \theta \) for \( \lambda \) odd, let \( \bar{x} = (\lambda + 1, 1) \) and \( \bar{y} = (\lambda, 1) \).

In case \( ((0, 0), (0, 1)) \in \theta \), let \( \bar{x} = (1, 0) \) and \( \bar{y} = (1, 1) \).

In case \( ((1, 0), (1, 1)) \in \theta \), let \( \bar{x} = (2, 1) \) and \( \bar{y} = (2, 0) \).

Then, in all cases, \( (\bar{x} \land u, \bar{y} \land u), (\bar{x} \land \bar{y}, \bar{x} \land u), (\bar{x} \land (0, 1), (0, 1)) \) and \( (\bar{y} \land (1, 1), (1, 1)) \) belong to \( \theta \).
It follows from this (see Taylor [8], Lemma 3.4, or Banaschewski and Nelson [1], Lemma 5) that $S_n$ is pure irreducible for all (finite or infinite) $n > 0$.

As mentioned above, Taylor has proved that every equationally compact semilattice is a retract of a product of finite semilattices. His short proof uses some rather complicated results, including the one of Numakura [6] that every totally disconnected topological semigroup is an inverse limit of finite discrete ones. We close by giving a modification of Taylor's proof which is straightforward and self-contained, and does not use this result of Numakura. The following proposition is essentially a corollary to Theorem 3.1 of Pacholski and Weglorz [7] (see also Taylor [9], Lemma 7.1); the present proof uses algebraic, rather than model-theoretic, techniques.

**Proposition.** If $A$ is a compact Hausdorff topological algebra, and if $\mathcal{C}$ is a down-directed set of closed congruences on $A$ with

$$\bigcap \mathcal{C} = \Lambda_A = \{(a, a) \mid a \in A\},$$

then the embedding $A \to \prod A/\theta \ (\theta \in \mathcal{C})$ is pure (and hence is retractable).

**Proof.** Let $A[X]$ be, as usual, the free extension over $A$ by the set $X$ in some convenient equational class containing $A$. For each $g \in A^X$, let $\tilde{g} : A[X] \to A$ be the homomorphism over $A$ extending $g$. Then, for each $p \in A[X]$, the map $\tilde{p} : A^X \to A$, given by $\tilde{p}(g) = \tilde{g}(p)$, is continuous with respect to the product topology on $A^X$. Thus, for each $(p, q) \in A[X]^2$, if $\theta$ is a closed subset of $A^2$, then

$$\{g \in A^X \mid (\tilde{g}(p), \tilde{g}(q)) \in \theta\} = \{g \in A^X \mid (\tilde{p}(g), \tilde{q}(g)) \in \theta\}$$

is a closed subset of $A^X$.

Now suppose that $\Sigma \subseteq A[X]^2$ is a finite subset which is contained in the kernel of some homomorphism $A[X] \to \prod A/\theta \ (\theta \in \mathcal{C})$ over $A \to \prod A/\theta$. For each $\theta \in \mathcal{C}$, let

$$S_\theta = \{g \in A^X \mid \tilde{g}^2(\Sigma) \subseteq \theta\};$$

then the $S_\theta$ form a down-directed collection of non-empty closed subsets of $A^X$. Since $A$, and hence also $A^X$, is compact, there exists $g \in \bigcap S_\theta$. But $\bigcap \mathcal{C} = \Lambda_A$, and thus $\Sigma \subseteq \text{Ker}\tilde{g}$. This proves that the embedding in question is pure.

**Corollary 1.** If $A$ is a compact algebra in which the closed congruences $\theta$ of finite index (i.e. with $A/\theta$ finite) separate the points of $A$, then $A$ is the (algebraic) retract of a product of finite algebras (each of which is a quotient of $A$).

**Proof.** In any topological algebra, the set of closed congruences of finite index forms a down-directed set (since there is an embedding $A/\theta_1 \cap \theta_2 \to A/\theta_1 \times A/\theta_2$).
Corollary 2. Every equationally compact semilattice is a retract of a product of finite ones.

Proof. Bulman-Fleming [2] has shown that every equationally compact semilattice is the retract of its closure $\overline{S}$ in $2^S$. The closure $\overline{S}$, being a compact subspace of $2^S$, obviously satisfies the hypotheses of Corollary 1.

One final comment. $S_n$ is the underlying $\wedge$-semilattice of a complete lattice, and we have seen that, despite this, $S_n$ (for $n$ infinite) does not have an equationally compact hull. However, since, by the Grätzer-Lakser characterization, the underlying $\wedge$-semilattice of the lattice of all ideals of any lattice is an equationally compact semilattice, and since the embedding of every distributive lattice into its ideal lattice is pure (Nelson [5]), and hence also pure as a semilattice homomorphism, it follows that any semilattice which is the underlying $\wedge$-semilattice of a distributive lattice has an equationally compact hull in the class of all semilattices.

REFERENCES


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