On double integrals and Fourier series

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In this paper we present some results concerning the Riemann-Stieltjes, Lebesgue and Titchmarsh integrals on a given rectangle, and Fourier series of two variables.

1. We shall introduce fundamental definitions and notation used in the sequel.

Let \( \varphi(x, y) \) be a function defined in the rectangle \( R = [a, b; c, d] \), and let

\[
P = \left\{ \begin{array}{l}
a = x_1 < x_2 < \ldots < x_i < \ldots < x_m < x_{m+1} = b \\
c = y_1 < y_2 < \ldots < y_j < \ldots < y_n < y_{n+1} = d
\end{array} \right\}
\]

be a partition of \( R \). Write

\[
\Delta \varphi(x_i, y_j) = \varphi(x_i, y_j) - \varphi(x_{i+1}, y_j) - \varphi(x_i, y_{j+1}) + \varphi(x_{i+1}, y_{j+1}).
\]

If \( \Delta \varphi(x_i, y_j) \geq 0 \) \( (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n) \) for any partition \( P \), we say that \( \varphi(x, y) \) satisfies the condition \( \Omega \) in \( R \).

The function \( \varphi(x, y) \) is said to be of finite variation in \( R \) if the sums

\[
V(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} |\Delta \varphi(x_i, y_j)|
\]

remain uniformly bounded for all possible \( P \). The upper bound \( V \) of all \( V(P) \), called the variation of \( \varphi \) in \( R \), will be denoted by

\[
V = \int_a^b \int_c^d |d \varphi(x, y)|.
\]

Of course, \( \varphi \) is of finite variation in \( R \) if it satisfies the condition \( \Omega \) in \( R \); in this case

\[
\int_a^b \int_c^d |d \varphi(x, y)| = \varphi(a, c) - \varphi(b, c) - \varphi(a, d) + \varphi(b, d).
\]

It is also well-known ([9], pp. 470, 471) that any function \( \varphi(x, y) \) of finite variation in \( R \) can be represented in the form

\[
\varphi(x, y) = -\varphi(a, c) + \varphi(x, c) + \varphi(a, y) + V_1^+(x, y) - V_1^-(x, y)
\]
or

\begin{align*}
\varphi(x, y) = -\varphi(b, d) + \varphi(x, d) + \varphi(b, y) + V_2^+(x, y) - V_2^-(x, y),
\end{align*}

where the functions $V_i^+(x, y)$ and $V_i^-(x, y)$ are non-negative, non-decreasing for $i = 1$ and non-increasing for $i = 2$ with respect to each variable separately and satisfy the condition $\Omega$ in $R$. Moreover,

\begin{align*}
V_1^+(x, c) = V_1^-(x, c) = V_1^+(a, y) = V_1^-(a, y) = 0
\end{align*}

and

\begin{align*}
V_2^+(x, d) = V_2^-(x, d) = V_2^+(b, y) = V_2^-(b, y) = 0
\end{align*}

for $(x, y) \in R$. The functions $V_i^\pm(x, y)$ ($i = 1, 2$) are continuous in $R$ if $\varphi(x, y)$ is continuous therein.

Let $f(x, y)$ and $g(x, y)$ be two functions defined in the rectangle $R = [a, b; c, d]$. Take a normal sequence of partitions

\begin{align*}
P_k = \begin{cases}
  a = x^{(k)}_1 < x^{(k)}_2 < \ldots < x^{(k)}_m < x^{(k)}_{m+1} = b \\
  c = y^{(k)}_1 < y^{(k)}_2 < \ldots < y^{(k)}_n < y^{(k)}_{n+1} = d
\end{cases}
\end{align*}

and points $(\xi_{i,j}^{(k)}, \eta_{j,i}^{(k)})$ and $(\xi_{i,j}^{(k)}, \eta_{j,i}^{(k)})$ in rectangles $[x^{(k)}_i, x^{(k)}_{i+1}; y^{(k)}_j, y^{(k)}_{j+1}]$ ($i = 1, 2, \ldots, m_k; j = 1, 2, \ldots, n_k$). Write

\begin{align*}
S(P_k) = \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} f(\xi_{i,j}^{(k)}, \eta_{j,i}^{(k)}) \Delta g(x^{(k)}_i, y^{(k)}_j)
\end{align*}

and

\begin{align*}
\overline{S}(P_k) = \sum_{i=1}^{m_k} \sum_{j=1}^{n_k} f(\xi_{i,j}^{(k)}, \eta_{j,i}^{(k)}) \Delta g(x^{(k)}_i, y^{(k)}_j).
\end{align*}

If a finite limit

\begin{align*}
\lim_{k \to \infty} S(P_k) = I \quad [\lim_{k \to \infty} \overline{S}(P_k) = \overline{I}]
\end{align*}

exists for any normal sequence $\{P_k\}$ and arbitrary points $(\xi_{i,i}^{(k)}, \eta_{i,i}^{(k)})$ [(\overline{\xi}_{i,i}^{(k)}, \overline{\eta}_{i,i}^{(k)})], we say that $f$ is $S$-integrable [\overline{S}-integrable] with respect to $g$ on $R$; the number $I$ [\overline{I}] is called the Riemann-Stieltjes integral of $f$ and denoted by

\begin{align*}
(\text{S}) \int_a^b \int_c^d f(x, y) \, dg(x, y) \quad \text{[\overline{S}] \int_a^b \int_c^d f(x, y) \, dg(x, y)}.
\end{align*}

The Lebesgue integral of the Lebesgue-integrable (L-integrable) function $f(x, y)$ on $R$ will be denoted by

\begin{align*}
(\text{L}) \int_a^b \int_c^d f(x, y) \, dxdy \quad \text{or} \quad (\text{L}) \int_R f(x, y) \, dxdy.
\end{align*}
An analogical terminology will be used in the one-dimensional case. For example, we say that \( f(x) \) is \( S \)-integrable with respect to \( g(x) \) in the interval \( \langle a, b \rangle \) if the simple Stieltjes integral

\[
(S) \int_{a}^{b} f(x) \, dg(x)
\]

is finite. Evidently, the concepts of \( S \) and \( \bar{S} \)-integrability now coincide.

Finally, we define a double integral of Titchmarsh’s type (cf. [1], pp. 585-587).

Let a function \( f(x, y) \) be plane-measurable with respect to a Lebesgue measure \( \mu \) in the rectangle \( R = [a, b; c, d] \) and let

\[
\mu \{ (x, y) \in R : |f(x, y)| > n \} = o(1/n) \quad \text{as} \quad n \to \infty.
\]

Write

\[
[f(x, y)]_n = \begin{cases} 
 f(x, y) & \text{when} \quad |f(x, y)| \leq n, \\
 0 & \text{when} \quad |f(x, y)| > n.
\end{cases}
\]

If the Lebesgue integrals

\[
(L) \int_{a}^{s} \int_{c}^{t} [f(x, y)]_n \, dx \, dy, \quad (s, t) \in R,
\]

tend uniformly to a limit \( F(s, t) \) (as \( n \to \infty \)) in \( R \), we say that \( f \) is \( T \)-integrable on \( R \); we denote the value \( F(b, d) \), termed the \( T \)-integral of \( f \), by

\[
(T) \int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy.
\]

We assume that the last integral equals zero in the case \( b = a \) or \( d = c \). Also, by definition,

\[
(T) \int_{b}^{a} \int_{c}^{d} f(x, y) \, dx \, dy = -(T) \int_{a}^{b} \int_{c}^{d} f(x, y) \, dx \, dy = (T) \int_{a}^{b} \int_{d}^{c} f(x, y) \, dx \, dy
\]

and

\[
(T) \int_{b}^{a} \int_{c}^{d} f(x, y) \, dx \, dy = (T) \int_{b}^{a} \int_{d}^{c} f(x, y) \, dx \, dy.
\]

All \( L \)-integrable functions are \( T \)-integrable; the converse is false (see [2], § 3). A \( T \)-integral is homogeneous and additive.

For the sake of brevity, we shall write \( f \in L(R) \) or \( f \in T(R) \) etc. if \( f \) is \( L \)-integrable or \( T \)-integrable etc. on \( R \), respectively. If \( f(x, y) \) is \( L \)-inte-
grable [T-integrable] with $p$th power on the square $Q = [-\pi, \pi; -\pi, \pi]$ and $2\pi$-periodic in each variable, we write $f \in L^p_{2\pi} \ [f \in T^p_{2\pi}]$. The symbols $L^p_{2\pi}$ and $T^p_{2\pi}$ will be used instead of $L^1_{2\pi}$ and $T^1_{2\pi}$. Also ($S$), ($\overline{S}$), ($L$), ($T$) before integrals will sometimes be omitted.

Let $(u_{m,n})$ and $(v_{m,n})$, $m, n = 1, 2, \ldots$, be two double sequences. We shall write

$$u_{m,n} = O(v_{m,n}) \quad \text{as} \quad (m, n) \to \infty,$$

if the quotient $u_{m,n}/v_{m,n}$ is bounded for all pairs $(m, n)$.

2. We shall now give some results concerning the $S$ or $\overline{S}$-integrability, and the relations between integrals on the rectangle $R = [a, b; c, d]$.

Start with the well-known theorem ([3], pp. 515-516).

2.1. A continuous function $f(x, y)$ is $S$-integrable on $R$ with respect to a function $g(x, y)$ of finite variation in $R$.

The following two-dimensional analogue of integration by parts can also easily be deduced (cf. [3], p. 609).

2.2. Let $f(x, y)$ be $\overline{S}$-integrable with respect to $g(x, y)$ on rectangle $R$, in the interval $\langle a, b \rangle$ for $y = c, y = d$, and in the interval $\langle c, d \rangle$ for $x = a, x = b$. Then $g(x, y)$ is $\overline{S}$-integrable with respect to $f(x, y)$ on $R$ and

$$\tag{4} \overline{S} \int_a^b \int_c^d g(x, y) df(x, y) = \overline{S} \int_a^b \int_c^d f(x, y) dg(x, y) + \int_a^b f(x, c) dg(x, c) -$$

$$- \int_a^b f(x, d) dg(x, d) + \int_c^d f(a, y) dg(a, y) - \int_c^d f(b, y) dg(b, y) + D,$$

where

$$D = f(a, c) g(a, c) - f(b, c) g(b, c) - f(a, d) g(a, d) + f(b, d) g(b, d).$$

In particular, we have

2.3. Suppose that $f(x, y)$ is continuous and $g(x, y)$ is of finite variation in $R$, and that $f(x, y)$ is $S$-integrable with respect to $g(x, y)$ in the intervals $\langle a, b \rangle$ (for $y = c, y = d$) and $\langle c, d \rangle$ (for $x = a, x = b$). Then $g(x, y)$ is $\overline{S}$-integrable with respect to $f(x, y)$ on $R$.

By 2.1, 2.2 and representation (2) [(2')] we obtain (cf. [3], p. 516).

2.4. Let $f(x, y)$ be continuous in $R$ and let $g(x, y)$ together with $f(x, y)$ have finite variations in $R$. If, moreover, $g(x, c) \ [g(x, d)]$ and $g(a, y) \ [g(b, y)]$ are of finite variation or continuous in the intervals $\langle a, b \rangle$ and $\langle c, d \rangle$, respectively, then $g(x, y)$ is $S$-integrable with respect to $f(x, y)$ on $R$; in particular, $g(x, y)$ is Riemann-integrable on $R$ (take $f(x, y) = xy$).
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From 2.4 and a suitable test of integrability the next theorem follows immediately (see [6], p. 233).

2.5. Suppose that \( f(x, y) \) is \( L \)-integrable on \( R \) and \( g(x, y) \) satisfies the same assumptions as in 2.4. Then \( g(x, y) \) is \( S \)-integrable with respect to

\[
F(x, y) = (L) \int_a^z \int_y^c f(s, t) \, ds \, dt \quad \text{on} \quad R, \quad fg \in L(R)
\]

and

\[
(S) \int_a^b \int_c^d g(x, y) \, dF(x, y) = (L) \int_a^b \int_c^d f(x, y) g(x, y) \, dx \, dy.
\]

This result remains true for \( g(x, y) \) continuous in \( R \).

Applying 2.5 and 2.2, we get (see [2], § 3)

2.6. Let \( f(x, y) \) be \( T \)-integrable on \( R \), and let

\[
F(x, y) = (T) \int_a^z \int_y^c f(s, t) \, ds \, dt \quad \text{for} \quad (x, y) \in R.
\]

Suppose that \( g(x, y) \) is of finite variation in \( R \) and that the variations

\[
\int_a^b |dg(x, v)|, \quad \int_c^d |dg(u, y)|
\]

are uniformly bounded for all \((u, v) \in R\). Then \( g(x, y) \) is \( S \)-integrable with respect to \( F(x, y) \) on \( R \), \( fg \in T(R) \) and

\[
(S) \int_a^b \int_c^d g(x, y) \, dF(x, y) = (T) \int_a^b \int_c^d f(x, y) g(x, y) \, dx \, dy.
\]

In particular our assertion holds if the function \( g(x, y) \) of finite variation in \( R \) is monotonic with respect to each variable separately in \( R \).

Denote by \( C(R) \) the Banach space of all functions \( f(x, y) \) continuous in \( R \), with norm \( \|f\| = \max_{R} |f(x, y)| \).

Arguing as in [5], pp. 167-170, we can state the following analogue of F. Riesz's theorem.

2.7. A necessary and sufficient condition for \( \Phi(f) \) to be a linear functional in \( C(R) \) is that

\[
\Phi(f) = (S) \int_a^b \int_c^d f(x, y) \, dg(x, y),
\]

where \( g(x, y) \) is of finite variation in \( R \).
3. In this section we present two-dimensional analogues of the theorems given in [4], pp. 79-81, and [7], pp. 211-213.

For the sake of brevity we shall write \( \int_a^b \int_c^d d\gamma(s,t) \) instead of \( \gamma(a,c) - \gamma(x,c) - \gamma(a,y) + \gamma(x,y) \).

3.1. Let \( \alpha(x,y) \) and \( \beta(x,y) \) be defined in the rectangle \( R = [a, b; c, d] \) and let \( \int_a^b \int_c^d d\beta(s,t) > 0 \) for \( a < x \leq b, \ c < y \leq d \). Write

\[
M_1 = \inf_{(x,y)} \int_a^b \int_c^d d\alpha(s,t), \quad M_2 = \sup_{(x,y)} \int_a^b \int_c^d d\beta(s,t)
\]

for \( a < x \leq b, \ c < y \leq d \). Suppose that \( \varphi(x,y) \) is non-negative, non-increasing with respect to each variable separately, satisfies the condition \( \Omega \) in \( R \) and is \( \mathcal{S} \)-integrable with respect to \( \alpha(x,y) \) and \( \beta(x,y) \) on this rectangle. Then

\[
M_1 I(\beta) \leq I(\alpha) \leq M_2 I(\beta),
\]

where

\[
I(\gamma) = (\mathcal{S}) \int_a^b \int_c^d \varphi(x,y) d\gamma(x,y).
\]

Proof. Given partition (1) let

\[
\mathcal{S}(P; \alpha) = \sum_{i=1}^m \sum_{j=1}^n \varphi(\xi_i, \eta_j) \Delta \alpha(\xi_i, \eta_j),
\]

where

\[
\xi_1 = x_1 = a, \quad \eta_1 = y_1 = c, \quad \xi_m = x_{m+1} = b, \quad \eta_n = y_{n+1} = d, \quad (\xi_i, \eta_j) \in [x_i, x_{i+1}; y_j, y_{j+1}] (i = 2, 3, \ldots, m-1; j = 2, 3, \ldots, n-1).
\]

By Abel's transformation ([7], p. 210, (2)),

\[
\mathcal{S}(P; \alpha) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \int_a^{x_{i+1}} \int_c^{y_{j+1}} d\alpha(s,t) \cdot \int_{\xi_i}^{\xi_{i+1}} \int_{\eta_j}^{\eta_{j+1}} d\varphi(s,t) +
\]

\[
+ \sum_{i=1}^{m-1} \int_a^b \int_c^d d\alpha(s,t) \cdot \int_{\xi_{i+1}}^{\xi_i} d\varphi(s,d) + \sum_{j=1}^{n-1} \int_a^b \int_c^d d\alpha(s,t) \cdot \int_{\eta_{j+1}}^{\eta_j} d\varphi(b,t) +
\]

\[
+ \int_a^b \int_c^d d\alpha(s,t) \cdot \varphi(b,d);
\]

whence

\[
M_1 \mathcal{S}(P; \beta) \leq \mathcal{S}(P; \alpha) \leq M_2 \mathcal{S}(P; \beta).
\]
If \( \{P_k\} \) denotes the normal sequence of partitions \((3)\), \( \overline{S}(P_k; \alpha) \) and \( \overline{S}(P_k; \beta) \) are of form \((7)\) with suitable points \((\xi^{(k)}, \eta^{(k)})\), we have
\[
\lim_{k \to \infty} \overline{S}(P_k; \alpha) = (\overline{S}) \int_a^b \int_c^d \varphi(x, y) \, d\alpha(x, y),
\]
\[
\lim_{k \to \infty} \overline{S}(P_k; \beta) = (\overline{S}) \int_a^b \int_c^d \varphi(x, y) \, d\beta(x, y).
\]
Hence, by \((9)\) and \((8)\), inequality \((6)\) follows.

The following corollary is useful.

3.2. Let \( f(x, y) \) and \( g(x, y) \) be T-integrable on \( R = [a, b; c, d] \),
\[
a(x, y) = (T) \int_a^x \int_c^y f(s, t) \, ds \, dt, \quad \beta(x, y) = (T) \int_a^z \int_c^y g(s, t) \, ds \, dt > 0
\]
for \( a < x \leq b \), \( c < y \leq d \). Suppose that \( \varphi(x, y) \), non-negative and non-increasing in each variable, satisfies the condition \( \Delta \) in \( R \). Then, by \( 2.6 \) and \( 3.1 \),
\[
M_1(T) \int_a^b \int_c^d \varphi \, dx \, dy \leq (T) \int_a^z \int_c^y f(s, t) \, ds \, dt \leq M_2(T) \int_a^b \int_c^d \varphi \, dx \, dy,
\]
where \( M_1, M_2 \) denote the infimum and the supremum of the quotient
\[
(T) \int_a^x \int_c^y f(s, t) \, ds \, dt / (T) \int_a^z \int_c^y g(s, t) \, ds \, dt
\]
for \( a < x \leq b \), \( c < y \leq d \) (cf. \([7]\), p. 211).

Taking \( g(x, y) = 1 \) in \( 3.2 \), we get the estimate (cf. \([7]\), pp. 213-215)
\[
\left| (T) \int_a^x \int_c^y f(x, y) \varphi(x, y) \, dx \, dy \right| \leq M \int_a^z \int_c^y \varphi(x, y) \, dx \, dy,
\]
with
\[
M = \sup_{a < x \leq b \atop c < y \leq d} \left| \frac{1}{(x-a)(y-c)} (T) \int_a^z \int_c^y f(s, t) \, ds \, dt \right|.
\]
This is an analogue of Natanson’s lemma ([6], p. 245).

Now a generalization of \((10)\) will be given.

3.3. Let \( \varphi(x, y) \) satisfy the same assumptions as \( g(x, y) \) in \( 2.6 \). If \( f(x, y) \) is T-integrable on \( R \), then
\[
\left| (T) \int_a^z \int_c^y f(x, y) \varphi(x, y) \, dx \, dy \right|
\]
\[
\leq M \int_a^z \int_c^y \left[ A(x, y) + B(x) + C(x) + |\varphi(b, d)| \right] \, dx \, dy,
\]
where $M$ is defined by (11) and

$$A(x, y) = \begin{cases} \int_{x}^{b} \int_{y}^{d} |d\varphi(s, t)| & \text{for } a \leq x < b, \ c \leq y < d, \\ 0 & \text{otherwise}; \end{cases}$$

$$B(x) = \begin{cases} \int_{a}^{b} |d\varphi(s, d)| & \text{for } a \leq x < b, \\ 0 & \text{for } x = b; \end{cases}$$

$$C(x) = \begin{cases} \int_{c}^{d} |d\varphi(b, t)| & \text{for } c \leq y < d, \\ 0 & \text{for } y = d. \end{cases}$$

Proof. Write

$$I = (T) \int_{a}^{b} \int_{c}^{d} f(x, y)\varphi(x, y) \, dx \, dy,$$

and

$$F(x, y) = (T) \int_{a}^{b} \int_{c}^{d} f(s, t) \, ds \, dt \quad \text{for } (x, y) \in \mathbb{R}.$$

Applying (5) and (4), we have

$$I = (\bar{S}) \int_{a}^{b} \int_{c}^{d} \varphi(x, y) dF(x, y) = (\bar{S}) \int_{a}^{b} \int_{c}^{d} F(x, y) \, d\varphi(x, y) -$$

$$- \int_{a}^{b} F(x, d) \, d\varphi(x, d) - \int_{c}^{d} F(b, y) \, d\varphi(b, y) + F(b, d)\varphi(b, d),$$

whence

$$|I| \leq \int_{a}^{b} \int_{c}^{d} |F(x, y)| \, dA(x, y) - \int_{a}^{b} |F(x, d)| \, dB(x) - \int_{c}^{d} |F(b, y)| \, dC(x) +$$

$$+ |F(b, d)| \, |\varphi(b, d)| = I_1 - I_2 - I_3 + K.$$

In view of (11) and (4)

$$I_1 \leq M \int_{a}^{b} \int_{c}^{d} (x - a)(y - c) \, dA(x, y) = M \int_{a}^{b} \int_{c}^{d} A(x, y) \, dx \, dy.$$

Similarly, (11) and integration by parts leads to

$$-I_2 \leq M(d - c) \int_{a}^{b} (x - a) \, d(-B(x)) = M \int_{a}^{b} \int_{c}^{d} B(x) \, dx \, dy,$$

$$-I_3 \leq M(b - a) \int_{c}^{d} (y - c) \, d(-C(x)) = M \int_{a}^{b} \int_{c}^{d} C(x) \, dx \, dy.$$

Summing up the results, we obtain inequality (12).
Also, three variants of (12) can be established by taking

$$M = \sup_{\substack{a \leq x \leq b \cr c \leq y \leq d}} \left| \frac{1}{(b-x)(y-c)} \int_a^b \int_c^d f(s, t) \, ds \, dt \right|,$$

etc. instead of $M$ defined by (11).

4. Now we shall give two theorems of Lebesgue's type (cf. [6], pp. 240-242).

4.1. Suppose that the functions $g_n(x, y)$ of finite variations in $R = [a, b; c, d]$ are uniformly bounded in $R$ for $n = 1, 2, \ldots$ Moreover, let the variations

$$\int_a^b \int_c^d |dg_n(x, y)|, \quad \int_a^b |dg_n(x, d)|, \quad \int_c^d |dg_n(b, y)|$$

remain uniformly bounded for $n = 1, 2, \ldots$, and let

$$\lim_{n \to \infty} \left( \int_a^b \int_c^d g_n(x, y) \, dx \, dy \right) = 0$$

for each $(a, \gamma) \in R$. Then

$$\lim_{n \to \infty} \left( \int_a^b \int_c^d f(x, y) g_n(x, y) \, dx \, dy \right) = 0$$

for any $f \in T$-integrable on $R$.

Proof. Given $\varepsilon > 0$ there is a function $f_\varepsilon \in L$-integrable on $R$ such that

$$\left| \left( T \int_a^b \int_c^d f(s, t) \, ds \, dt - (L \int_a^b \int_c^d f_\varepsilon(s, t) \, ds \, dt) \right) \right| < \varepsilon$$

uniformly, with respect to $(x, y)$, in $R$. Write

$$F(x, y) = (T \int_a^b \int_c^d f(s, t) \, ds \, dt), \quad F_\varepsilon(x, y) = (L \int_a^b \int_c^d f_\varepsilon(s, t) \, ds \, dt),$$

$$H_n = (T \int_a^b \int_c^d g_n(s, t) \, ds \, dt), \quad I_n = (L \int_a^b \int_c^d f_\varepsilon(s, t) g_n(s, t) \, ds \, dt).$$

In view of 2.6 and 2.2,

$$|H_n - I_n| = \left| (\mathcal{S} \int_a^b \int_c^d g_n(x, y) \, dx \, dy) \right| \leq K \varepsilon,$$

where $K$ is a positive constant.
Since $I_n \to 0$ as $n \to \infty$, our result is established.

It is easy to see that if $g_n(x, y)$ are uniformly bounded and monotonic in each variable separately, and if they satisfy the condition $\Omega$ in $R$, the variation-restrictions on $g_n$ follow automatically.

The index $n$ in 4.1 can be replaced by the parameter $\sigma = (\xi, \eta)$, where $(\xi, \eta)$ belong to a set $E$ on the plane, with an accumulation point $\tau = (\xi_0, \eta_0)$. In this case, we consider $\sigma \to \tau$ instead of $n \to \infty$.

This parameter will be introduced in the formulation of the next theorems.

4.2. Suppose that the functions $K(s, t; \sigma)$ are non-negative, 2$\pi$-periodic, even and non-increasing in the square $Q_0 = [0, \pi; 0, \pi]$ with respect to each of the variables $s$ and $t$ separately, and that they satisfy the condition $\Omega$ in $Q_0$ for every fixed $\sigma \in E$. Let $\delta, x_0, c, d$ be fixed, $0 < \delta < \pi$, $-\pi + \delta \leq x_0 \leq \pi - \delta$, $-\pi \leq c < d \leq \pi$, and let the values $K(\delta/2, 0; \sigma)$ remain uniformly bounded for all $\sigma \in E$. Then, if the relation

$$
\lim_{\delta \to \tau} \int_{x_0 + \delta}^{x_0 + \delta + c} \int_{x_0 - \delta}^{x_0 + \delta} K(s - x, t - y; \sigma) \, ds \, dt = 0 \quad (\sigma \in E)
$$

holds uniformly in $x, y$, $x_0 - \delta/2 \leq x \leq x_0 + \delta/2$, $-\pi \leq y \leq \pi$, we have

$$(13) \quad \lim_{\delta \to \tau} (T) \int_{x_0 + \delta}^{x_0 + \delta + c} \int_{x_0 - \delta}^{x_0 + \delta} f(s, t) K(s - x, t - y; \sigma) \, ds \, dt = 0 \quad (\sigma \in E)$$

uniformly with respect to $x, y$ for any function $f \in T$-integrable on $[x_0 + \delta, x_0 + \delta; c, d]$.

Relation (13) for continuous or essentially bounded $f$ can be established similarly to that of [6], p. 241. In the general case ($f \in T$) we argue as in 4.1.

Analogical results hold for the integrals

$$
\int_{-\pi}^{\pi} \int_{a}^{b} \int_{x_0 - \delta}^{x_0 + \delta} f(s, t) K(s - x, t - y; \sigma) \, ds \, dt
$$

of the function $f(s, t) K(s - x, t - y; \sigma)$.

5. Suppose that the functions $\varphi(x, y; \sigma)$ are non-negative, and non-increasing with respect to $x$ and $y$ separately, and that they satisfy the condition $\Omega$ in the rectangle $R = [a, b; c, d]$ for any fixed $\sigma \in E$. Denote by $T^*(R)$ the class of all functions $f \in T(R)$ such that

$$
\lim_{h \to 0+} \lim_{k \to 0+} \int_{a}^{a+h} \int_{c}^{c+k} f(x, y) \, dx \, dy = f(a, c).
$$
We shall now give a theorem of Romanovski's type concerning the convergence of singular integrals

\[ I(\sigma; f) = (T) \int_a^b \int_c^d f(x, y) \varphi(x, y; \sigma) \, dx \, dy \]

as \( \sigma \to \tau \), for \( f \in T^\ast(R) \). Of course, these integrals exist by 2.6.

5.1. If

\[ \lim_{\sigma \to \tau} \int_a^\gamma \int_c^\delta \varphi(x, y; \sigma) \, dx \, dy = 1 \quad (\sigma \in E) \]

and

\[ \varlimsup_{\sigma \to \tau} \varphi(a, \gamma; \sigma) < \infty, \quad \varliminf_{\sigma \to \tau} \varphi(a, \gamma; \sigma) < \infty \quad (\sigma \in E), \]

with any fixed \( a, \gamma \) \( (a < a \leq b, \ c < \gamma \leq d) \), then

\[ \lim_{\sigma \to \tau} I(\sigma; f) = f(a, c) \quad (\sigma \in E) \]

for every \( f \in T^\ast(R) \).

Proof. Obviously, it is sufficient to show that

\[ U(\sigma; f) = (T) \int_a^b \int_c^d g(x, y) \varphi(x, y; \sigma) \, dx \, dy \to 0 \quad \text{as} \quad \sigma \to \tau, \]

where \( g(x, y) = f(x, y) - f(a, c) \).

Since \( f \in T^\ast(R) \), given \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that

\[ \left| \frac{1}{h \delta} (T) \int_{a+h}^{a+k} \int_{c}^{c+k} g(x, y) \, dx \, dy \right| < \varepsilon \]

for \( 0 < h, k \leq \delta \); whence, by (10),

\[ (14) \quad \left| (T) \int_a^{a+\delta} \int_c^{c+\delta} g(x, y) \varphi(x, y; \sigma) \, dx \, dy \right| \leq \varepsilon \int_a^b \int_c^d \varphi(x, y; \sigma) \, dx \, dy. \]

Write

\[ U(\sigma; f) = \left( \int_a^{a+\delta} \int_c^{c+\delta} + \int_{a+\delta}^b \int_c^d + \int_a^b \int_{c+\delta}^d \right) g(\sigma) \, dx \, dy = A_\sigma + B_\sigma + C_\sigma. \]

In view of (14),

\[ |A_\sigma| < 2 \varepsilon \]

for \( \sigma \) sufficiently near to \( \tau \).

Applying 4.1, we get

\[ \lim_{\sigma \to \tau} B_\sigma = 0 = \lim_{\sigma \to \tau} C_\sigma, \]

and our result is established.
We now formulate a theorem of Faddeev’s type ([6], p. 246) concerning the convergence of singular integrals

\[ J(\sigma; f) = (L) \int_a^b \int_c^d f(x, y) \psi(x, y; \sigma) \, dx \, dy, \]

where \( \psi(x, y; \sigma) \) are bounded (measurable) for any fixed \( \sigma \in E \), and together with \( f(x, y) \) subject to further restrictions.

Denote by \( L^*(R) \) the class of all \( f \in L(R) \) for which

\[ \lim_{h \to 0^+} \lim_{k \to 0^+} \frac{1}{hk} \int_a^{a+h} \int_c^{c+k} |f(x, y) - f(a, c)| \, dx \, dy = 0. \]

5.2. Suppose that

\[ \lim_{\sigma \to \gamma} \int_a^c \int_c^y \psi(x, y; \sigma) \, dx \, dy = 1 \]

and \( \psi(x, y; \sigma) \) remain uniformly bounded in the domain \([a, b; c, d] - [-a, a; c, y] \) for every fixed \( a, \gamma (a < a \leq b, c < \gamma \leq d), \sigma \in E \). If, moreover, there exist functions \( \mu(x, y; \sigma) \) non-negative, non-increasing in \( x \) and \( y \), satisfying the condition \( \Omega \) in \( R \), and such that

\[ |\psi(x, y; \sigma)| \leq \mu(x, y; \sigma) \]

for \( (x, y) \in R, \sigma \in E \), and

\[ \lim_{\sigma \to \gamma} \int_a^b \int_c^d \mu(x, y; \sigma) \, dx \, dy < \infty \quad (\sigma \in E), \]

then

\[ \lim_{\sigma \to \gamma} J(\sigma; f) = f(a, c) \quad (\sigma \in E) \]

for every \( f \in L^*(R) \).

6. In the theory of summability of double Fourier series the integrals

\[ V(x, y; \sigma, f) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(s, t) K(s - x, t - y, \sigma) \, ds \, dt \quad (\sigma \in E) \]

play a fundamental role, where the functions \( K(s, t; \sigma) \) are bounded (measurable), even and 2\( \pi \)-periodic in each variable \( x, y \) separately, and

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} K(s, t; \sigma) \, ds \, dt = 1 \quad (\sigma \in E). \]
Double integrals and Fourier series

These integrals, for \( f \in T_{2\pi} \), can be rewritten in the form

\[
V(x, y; \sigma, f) = (T) \int_0^\pi \int_0^\pi g(s, t) K(s, t; \sigma) \, ds \, dt,
\]

where

\[
g(s, t) = f(x+s, y+t) + f(x-s, y+t) + f(x+s, y-t) + f(x-s, y-t).
\]

Applying 5.1 and 5.2, we get the following theorem.

**6.1.** If \( 4K(s, t; \sigma) \) satisfies the same conditions as \( \varphi(s, t; \sigma) \) [\( \psi(s, t; \sigma) \)] in 5.1 [5.2], with \( a = c = 0 \), \( b = d = \pi \), and if

\[
\lim_{h \to 0^+} \frac{1}{hk} (T) \int_0^h \int_0^k g(s, t) \, ds \, dt = 4f(x, y) \quad (f \in T_{2\pi})
\]

\[
\left[ \lim_{h \to 0^+} \frac{1}{hk} (L) \int_0^h \int_0^k |g(s, t) - 4f(x, y)| \, ds \, dt = 0 \quad (f \in L_{2\pi}) \right],
\]

we have

\[
\lim_{\sigma \to \pi} V(x, y; \sigma, f) = f(x, y).
\]

The above-mentioned and the following statements refer to the restricted summability of double Fourier series.

**6.2.** Let \( K(s, t; \sigma) \) satisfy all the assumptions of 4.2 and let \( K(\delta/2, 0; \sigma) \), \( K(0, \delta/2; \sigma) \) be uniformly bounded in \( E \), for any fixed \( \delta > 0 \) as small as we please. Suppose that the relations

\[
\lim_{\sigma \to \pi} \int_{x_0+\delta}^{\pi} \int_{-\pi}^{\pi} K(s-x, t-y; \sigma) \, ds \, dt = 0 \quad (\sigma \in E)
\]

and

\[
\lim_{\sigma \to -\pi} \int_{x_0-\delta}^{\pi} \int_{-\pi}^{\pi} K(s-x, t-y; \sigma) \, ds \, dt = 0 \quad (\sigma \in E)
\]

hold uniformly in \( x, y \), \( x_0 - \delta/2 \leq x \leq x_0 + \delta/2 \), \( -\pi \leq y \leq \pi \), for every fixed \( \delta \), \( 0 < \delta \leq \pi \pm x_0 \). Suppose that the relations

\[
\lim_{\sigma \to -\pi} \int_{-\pi}^{\pi} \int_{x_0+\delta}^{\pi} K(s-x, t-y; \sigma) \, ds \, dt = 0 \quad (\sigma \in E)
\]

and

\[
\lim_{\sigma \to -\pi} \int_{-\pi}^{\pi} \int_{x_0-\delta}^{\pi} K(s-x, t-y; \sigma) \, ds \, dt = 0 \quad (\sigma \in E)
\]
hold uniformly in \( x, y, -\pi \leq x \leq \pi, y_0 - \delta/2 \leq y \leq y_0 + \delta/2 \) \((y_0 = \text{const})\) for every fixed \( \delta, 0 < \delta \leq \pi \pm y_0 \). Denote by \( Z \) the set of points \((x, y; \sigma), \sigma \in E\), such that the product \((x-x_0)(y-y_0)K(0, 0; \sigma)\) remains bounded. Then the relation

\[
\lim_{h \to 0} \frac{1}{hk}(T) \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} f(s, t) ds dt = f(x_0, y_0) \quad (f \in T_{\mathbb{R}})
\]

implies

\[
\lim V(x, y; \sigma, f) = f(x_0, y_0) \quad (\sigma \in E)
\]
as \((x, y; \sigma) \to (x_0, y_0; \tau)\) on \( Z \).

The proof is similar to that of 3.1, [8]. Namely, given \( \varepsilon > 0 \) we choose a \( \delta > 0 \) such that

\[
\left| \frac{1}{hk}(T) \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} [f(s, t) - f(x_0, y_0)] ds dt \right| < \varepsilon
\]

for \(|h| \leq \delta, |k| \leq \delta\). Write

\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} [f(s, t) - f(x_0, y_0)] K(s-x, t-y; \sigma) ds dt = (\int_G + \int_{\Gamma}) [f(s, t) - f(x_0, y_0)] K(s-x, t-y; \sigma) ds dt = I_1 + I_2,
\]

where

\[
G = [x_0 - \delta, x_0 + \delta; y_0 - \delta, y_0 + \delta], \quad \Gamma = [-\pi, \pi; -\pi, \pi] - G.
\]

Applying 3.2, 3.3 and 4.2, we get

\[
|I_1| + |I_2| \leq C\varepsilon, \quad C = \text{const,}
\]

for \((x, y; \sigma)\) sufficiently near to \((x_0, y_0; \tau)\). The proof is thus finished. An analogous statement of Faddeev's type can also be given.

We note that theorem (3.1) of [10], p. 309 \((m = 2)\) follows at once from 6.1 and 6.2 (see also [9], p. 497).

7. Write

\[
\Delta^{(r)}_{h, k} f(x, y) = \sum_{\mu=0}^{r} \sum_{\nu=0}^{r} (-1)^{2r-\mu-\nu} \binom{r}{\mu} \binom{r}{\nu} f(x + \mu h, y + \nu k),
\]

\[
\Delta^{(r, 1)}_{h} f(x, y) = \sum_{\mu=0}^{r} (-1)^{r-\mu} \binom{r}{\mu} f(x + \mu h, y),
\]

\[
\Delta^{(r, 2)}_{h} f(x, y) = \sum_{\nu=0}^{r} (-1)^{r-\nu} \binom{r}{\nu} f(x, y + \nu k).
\]
The class of all functions \( f \in L^2_{2\pi} \) such that

\[
\left\{ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left| A_{k}^{(r)} f(x, y) \right|^2 \, dx \, dy \right\}^{1/2} \leq M h^{\alpha} k^{\beta} \quad (0 < \alpha, \beta \leq 1)
\]

for sufficiently small \( h, k > 0 \), \( M \) being a constant depending only on \( f \), will be denoted by \( A^{(r)}(\alpha, \beta) \).

The class of all \( f \in A^{(r)}(\alpha, \beta) \) for which

\[
\left\{ \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} A_{k}^{(r,1)} f(x, y) \, dy \right)^2 \, dx \right\}^{1/2} \leq M_1 h^{\alpha'} \quad (0 < \alpha' \leq 1)
\]

and

\[
\left\{ \int_{-\pi}^{\pi} \left( \int_{-\pi}^{\pi} A_{k}^{(r,2)} f(x, y) \, dx \right)^2 \, dy \right\}^{1/2} \leq M_2 k^{\beta'} \quad (0 < \beta' \leq 1)
\]

as \( h, k > 0 \) small enough, \( M_1, M_2 \) are positive constants, will be denoted by \( A^{(r)}(\alpha, \beta; \alpha', \beta') \).

Analogously we define the classes \( H^{(r)}(\alpha, \beta) \) and \( H^{(r)}(\alpha, \beta; \alpha', \beta') \) of certain 2\( \pi \)-periodic functions \( f(x, y) \) essentially bounded in the square \( Q = [-\pi, \pi; -\pi, \pi] \); we take the metrics \( \text{ess sup}_Q |\varphi(x, y)| \) and \( \text{ess sup}_Q |\psi(z)| \) in the above inequalities instead of \( \left\{ \int_{-\pi}^{\pi} \varphi^2(x, y) \, dx \, dy \right\}^{1/2} \) and \( \left\{ \int_{-\pi}^{\pi} \psi^2(x) \, dx \right\}^{1/2} \).

In the sequel, only \( r = 1 \) and \( r = 2 \) will be considered.

We shall now give some relations between the above-mentioned classes of functions and the classes of Fourier coefficients corresponding to them.

\textbf{7.1.} Let \( 0 < \alpha, \beta, \alpha', \beta' < 1 \). Suppose that \( a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n} \) \( (m, n \geq 1) \) are of the order \( O(m^{-\alpha + 1/2} n^{-\beta + 1/2}) \) as \( (m, n) \to \infty \) and that \( a_{m,0}, b_{m,0} \) and \( a_{0,n}, c_{0,n} \) are \( O(m^{-\alpha' + 1/2}) \) as \( m \to \infty \) and \( O(n^{-\beta' + 1/2}) \) as \( n \to \infty \), respectively. Then there is a function \( f \in A^{(1)}(\alpha, \beta; \alpha', \beta') \) having the double Fourier series ([9], pp. 435-446)

\[
\sum_{m,n=0}^{\infty} \lambda_{m,n} (a_{m,n} \cos mx \cos ny + b_{m,n} \sin mx \cos ny + c_{m,n} \cos mx \sin ny + d_{m,n} \sin mx \sin ny).
\]

\textbf{Proof.} By the Riesz-Fischer theorem, there is an \( f \in L^2_{2\pi} \) with Fourier series (15).

It is easy to verify that

\[
A_{2h,2k}^{(1)} f(x-h, y-k) = f(x-h, y-k) - f(x+h, y-k) - f(x-h, y+k) + f(x+h, y+k)
\]
has the Fourier series

\[
4 \sum_{m,n=1}^{\infty} \left( a_{m,n} \sin mx \sin ny - b_{m,n} \cos mx \sin ny -
- c_{m,n} \sin mx \cos ny + d_{m,n} \cos mx \cos ny \right) \sin mh \sin nk.
\]

In view of Parseval's identity ([9], p. 509),

\[
(16) \quad \sum_{m,n=1}^{\infty} \xi_{m,n}^2 (\sin mh \sin nk)^2 = \frac{1}{(4\pi)^2} \int \int_Q \{A_{m,n;k}^{(1)} f(x-h, y-k)\}^2 \, dx \, dy,
\]

where

\[
\xi_{m,n} = (a_{m,n}^2 + b_{m,n}^2 + c_{m,n}^2 + d_{m,n})^{1/2}.
\]

The left-hand side of (16) does not exceed

\[
\sum_{m=1}^{2^p-1} \sum_{n=1}^{2^q-1} \xi_{m,n}^2 (\sin mh \sin nk)^2 + \sum_{m=2^p}^{\infty} \sum_{n=2^q}^{\infty} \xi_{m,n}^2 +
+ \sum_{m=2^p}^{\infty} \sum_{n=1}^{2^q-1} \xi_{m,n}^2 \sin^2 nk + \sum_{m=1}^{2^p-1} \sum_{n=2^q}^{\infty} \xi_{m,n}^2 \sin^2 nk = S_1 + S_2 + S_3 + S_4.
\]

By hypothesis

\[
(17) \quad \sum_{m=\mu}^{\infty} \sum_{n=\nu}^{\infty} \xi_{m,n}^2 \leq A \frac{1}{\mu^{2a} \nu^{2b}}, \quad A = \text{const};
\]

whence

\[
\sum_{m=\mu}^{2^p-1} \sum_{n=\nu}^{2^q-1} (mn \xi_{m,n})^2 \leq 16 A \mu^{2(1-a)} \nu^{2(1-b)}.
\]

Therefore

\[
S_1 \leq 16 A (hk)^2 \sum_{i=1}^{p} \sum_{j=1}^{q} 2^{2(1-a)(i-1)} 2^{2(1-b)(j-1)}.
\]

Suppose that \( h, k \) are positive and choose integers \( p, q \) such that

\[
(18) \quad 2^{-p} < h \leq 2^{-p+1}, \quad 2^{-q} < k \leq 2^{-q+1}.
\]

Then

\[
S_1 = 16 A (hk)^2 \frac{2^{2(1-a)p} - 1}{2^{2(1-a)} - 1} \cdot \frac{2^{2(1-b)q} - 1}{2^{2(1-b)} - 1} \leq A_1 h^{2a} k^{2b}, \quad A_1 = \text{const}.
\]

By (17) and (18),

\[
S_2 \leq A \cdot 2^{-2ap} \cdot 2^{-2bq} \leq A k^{2a} k^{2b}.
\]
Further, we have
\[ S_3 \leq k^2 \sum_{n=1}^{2^s-1} \left\{ n^2 \sum_{m=2^n}^{2^n} \varrho_{m,n}^2 \right\} \leq Bk^2 \sum_{n=1}^{2^s-1} \frac{1}{n^{2a-1}} \cdot \frac{1}{n^{2b}} \]
\[ \leq B_1 k^2 \cdot \frac{1}{2^{2a}} \cdot 2 \leq 4B_1 k^2 \cdot k^{2b}, \]
where \( B \) and \( B_1 \) are positive constants. A similar inequality holds for \( S_4 \).

Summing up the results, we get \( f \in A^{(0)}(\alpha, \beta) \).

The functions
\[ \frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) \, dy \quad \text{and} \quad \frac{1}{\pi} \int_{-\pi}^{\pi} f(x, y) \, dx \]
have the Fourier series
\[ \frac{a_{0,0}}{2} + \sum_{m=1}^{\infty} (a_{m,0} \cos mx + b_{m,0} \sin mx) \]
and
\[ \frac{a_{0,0}}{2} + \sum_{n=1}^{\infty} (a_{0,n} \cos ny + c_{0,n} \sin ny), \]
respectively. Hence, \( f \in A^{(0)}(\alpha, \beta; \alpha', \beta') \) (see [1], pp. 208-210).

Also, we can prove that if the numbers \( a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n} \) \((m, n \geq 0)\) satisfy the assumptions of 7.1 for \( a = \beta = \alpha' = \beta' = 1 \), then there exists a continuous \( f \in A^{(0)}(1, 1; 1, 1) \) with Fourier series (15).

Under the additional restriction on the coefficients, the following inverse result holds.

7.2. Let \( 0 < \alpha, \beta, \alpha', \beta' < 1 \). Suppose that (15) is the Fourier series of \( f \in A^{(0)}(\alpha, \beta) \) and that \( a_{m,n}^2 + b_{m,n}^2 + c_{m,n}^2 + d_{m,n}^2 \) is non-increasing in \( m, n \geq 1 \), separately. Then \( a_{m,n}, b_{m,n}, c_{m,n}, d_{m,n} \) \((m, n \geq 1)\) are \( O(m^{-\alpha+\frac{1}{2}}n^{-\beta+\frac{1}{2}}) \) as \( (m, n) \to \infty \). If, moreover, \( f \in A^{(0)}(\alpha, \beta; \alpha', \beta') \) and if \( a_{m,0}^2 + b_{m,0}^2 \) and \( a_{0,n}^2 + c_{0,n}^2 \) are non-increasing in \( m \) and \( n \), then \( a_{m,0}, b_{m,0}, a_{0,n}, c_{0,n} \) are \( O(m^{-\alpha'+\frac{1}{2}}) \) as \( m \to \infty \) and \( O(n^{-\beta'+\frac{1}{2}}) \) as \( n \to \infty \), respectively.

We prove the first part only (cf. [1], pp. 678-679). By formula (16),
\[ \sum_{m=p}^{q} \sum_{n=r}^{s} \varrho_{m,n}^2 (\sin mh \sin nk)^2 \leq C |h|^{2\alpha} |k|^{2\beta}, \]
where
\[ p = [(q+1)/2], \quad r = [(s+1)/2], \quad C = \text{const.} \]

Set \( h = \pi/2q \), \( k = \pi/2s \). Since
\[ \sin^2 m \frac{\pi}{2q} \geq \frac{1}{2}, \quad \sin^2 n \frac{\pi}{2s} \geq \frac{1}{2}, \]
\[ \frac{1}{2}, \quad \frac{1}{2} \]
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for \( p \leq m \leq q \) and \( r \leq n \leq s \), we have

\[
\frac{1}{2^s} \sum_{m=p}^{q} \sum_{n=r}^{s} e_{m,n} \leq C \left( \frac{\pi}{2q} \right)^{2a} \left( \frac{\pi}{2s} \right)^{2b}.
\]

From the last inequality our result follows immediately.

We remark that the class \( \Lambda^{(1)}(1,1) \) in 7.2 can be replaced by \( \Lambda^{(2)}(1,1) \).

Similar results hold for the classes \( H^{(r)}(a, \beta) \) and \( H^{(r)}(a, \beta; a', \beta') \) \((r = 1, 2)\). Namely,

7.3. If \( f(x, y) \) is expansible in series (15) with coefficients \( a_{m,n}, \ldots \) of the order \( O(m^{-\alpha}n^{-\beta+1}) \) \((0 < \alpha, \beta < 1)\) \([O(m^{-2}\alpha}\beta)]\), then \( f \in H^{(1)}(a, \beta) \) \([f \in H^{(2)}(1, 1)]\).

If, moreover, \( a_{m,0}, b_{m,0}, \ldots \), are \( O(m^{-\alpha+1}) \) \([O(m^{-2}\alpha}\beta)]\), \( O(n^{-\beta+1}) \) \([O(m^{-2}\alpha}\beta)]\), respectively, where \( 0 < \alpha', \beta' < 1 \), then \( f \in H^{(1)}(a, \beta; a', \beta') \) \([f \in H^{(2)}(1, 1; 1, 1)]\) (cf. [1], pp. 208-210).

7.4. Let \( f \in H^{(1)}(a, \beta) \) \((0 < \alpha, \beta \leq 1)\) \([f \in H^{(2)}(1, 1)]\) and let

\[
f(x, y) = \sum_{m=n=0}^{\infty} \lambda_{m,n} a_{m,n} \cos mx \cos ny
\]

for all \((x, y)\), where \( a_{m,n} \) are non-negative and non-increasing in \( m, n \geq 1 \).

Then

\[
a_{m,n} = O(m^{-\alpha+1}n^{-\beta+1}) \quad [a_{m,n} = O(m^{-2}\alpha}\beta)] \quad \text{as} \quad (m, n) \to \infty.
\]

We obtain this statement at once from the identity

\[
A_{2h,2k}^{(2)} f(x-2h, y-2k) = 16 \sum_{m,n=1}^{\infty} a_{m,n} \cos mx \cos ny (\sin mh \sin nk)^2,
\]

taking \( x = y = 0 \) and arguing as in [1], p. 678.

7.5. Suppose that the function \( f \in H^{(1)}(a, \beta) \) \((0 < \alpha, \beta \leq 1)\) \([f \in H^{(2)}(1, 1)]\) has the Fourier series

\[
\sum_{m,n=1}^{\infty} d_{m,n} \sin mx \sin ny
\]

with coefficients \( d_{m,n} \geq 0 \) such that \( d_{m,n}/mn \) is non-increasing in \( m \geq 1 \) and \( n \geq 1 \), separately. Then

\[
d_{m,n} = O(m^{-\alpha+1}n^{-\beta+1}) \quad [d_{m,n} = O(m^{-2}\alpha}\beta)] \quad \text{as} \quad (m, n) \to \infty.
\]

To prove 7.5 we observe that \( A_{2h,2k}^{(2)} f(x-2h, n-2k) \) has the Fourier series

\[
16 \sum_{m,n=1}^{\infty} d_{m,n} \sin mx \sin ny (\sin mh \sin nk)^2.
\]
Hence ([9], pp. 514-516)

\[
\int_0^{2h} \int_0^{2k} A_{2h,2k}(x-2h, y-2k) \, dx \, dy = 64 \sum_{m,n=1}^{\infty} \frac{d_{m,n}}{mn} (\sin mh \sin nk)^4.
\]

Taking \( h = \pi/2m \), \( k = \pi/2n \) and reasoning as in [1], pp. 678-679, we shall establish our result.

References


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