NON-SIDON SETS IN THE SUPPORT
OF A FOURIER-STIELTJES TRANSFORM

BY

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1. Introduction. Several years ago, Hartman and Ryll-Nardzewski ([4], Problem 711) asked whether there exists a continuous measure $\mu$ on the circle group $T$ such that, for some $\varepsilon > 0$, $\{n: |\hat{\mu}(n)| > \varepsilon\} = B(\mu, \varepsilon)$ is not a Sidon set. An affirmative answer was first provided by Kaufman [7]. Kaufman used an elegant category argument to produce a class of examples. Later, the present author showed in [3] that if $\mu$ were concentrated on a Kronecker set (see [9] for a definition), then $B(\mu, \varepsilon)$ contained arbitrarily long arithmetic progressions for all $0 < \varepsilon < \|\mu\|$. A different class of examples was provided by Ramirez [8] who showed that Riesz products $\mu$ had the property that $B(\mu, \varepsilon)$ was not a Sidon set for $0 < \varepsilon \leq \limsup |\hat{\mu}(n)|^2$. Izuchi studies in [5] the class $\mathcal{S}$ of measures $\mu$ on an LCA group $G$ such that $B(\mu, \varepsilon)$ is a Sidon set for all $\varepsilon > 0$. He shows, in particular, that $\mathcal{S}$ is an $L$-ideal [12] of measures, and that $\mathcal{S}$ contains (for compact $G$, of course) the measures $M_\sigma(G)$ such that $\hat{\mu}$ vanishes at infinity on the dual $\Gamma$ of $G$.

This note improves the method of [3] to obtain the observation of [8]: $B(\mu, \varepsilon)$ contains arbitrarily “large squares” (see below) if $\hat{\mu}$ does not vanish at infinity on $\Gamma$. An (obvious modification of the) argument of Salinger and Varopoulos ([10], proof of Theorem 3) shows that if $B(\mu, \varepsilon)$ contains arbitrarily large squares, then $B(\mu, \varepsilon)$ is a set of analyticity, and so is not a Sidon set. (That $B(\mu, \varepsilon)$ is not Sidon also follows from [6], p. 54.) Thus, the class $\mathcal{S}$ of measures studied in [5] coincides with $M_\sigma(G)$ for compact $G$. The result of this note may also be considered as a partial converse to Drury’s theorem [2]. For another related result, see Blei [1].

We now define our terms and state our theorems.

Definition. An $n$-square in the LCA group $\Gamma$ is a set of the form $AB$, where $A, B$ are subsets of $\Gamma$ of cardinality $n$. A set $X \subseteq \Gamma$ contains arbitrarily large squares if $X$ contains an $n$-square for $n = 1, 2, \ldots$ (The reader will have noted that $\Gamma$ is written as a multiplicative group. This (unusual) convention will simplify the notation later on.)
The reader should be warned that an \( n \)-square may not have cardinality \( n^2 \). However, Salinger and Varopoulos [10] show that if \( X \) contains arbitrarily large squares, then \( X \) contains \( n \)-squares which do have \( n^2 \) elements for \( n = 1, 2, \ldots \).

For a regular Borel measure \( \mu \) on the LCA group \( G \) (with dual \( \Gamma \)) and \( \varepsilon > 0 \), we define \( E(\mu, \varepsilon) \) by
\[
E(\mu, \varepsilon) = \{ \gamma \in \Gamma : |\hat{\mu}(\gamma)| > \varepsilon \},
\]
where \( \hat{\mu} \) denotes the Fourier-Stieltjes transform of \( \mu \).

In this note we prove

**Theorem 1.** Let \( G \) be a non-discrete LCA group with dual \( \Gamma \). Let \( \mu \) be a regular Borel measure on \( G \) such that the Fourier-Stieltjes transform \( \hat{\mu} \) of \( \mu \) does not vanish at infinity on \( \Gamma \). Then there exists \( \varepsilon > 0 \) such that
\[
E(\mu, \varepsilon) = \{ \gamma \in \Gamma : |\hat{\mu}(\gamma)| \geq \varepsilon \}
\]
contains arbitrarily large squares.

**Corollary 1.** If \( G, \Gamma, \mu \) and \( \varepsilon \) satisfy the hypotheses and conclusion of Theorem 1, then \( E(\mu, \varepsilon) \) is a set of analyticity.

**Corollary 2.** Let \( G \) be an infinite LCA group with dual \( \Gamma \). Let \( \mu \) be a regular Borel measure on \( G \). If \( E(\mu, \varepsilon) \) is a Sidon set for all \( \varepsilon > 0 \), then \( \hat{\mu} \) vanishes at infinity on \( \Gamma \).

**2. Proofs.** Corollary 2 follows at once from Corollary 1, and Corollary 1 follows from (the proof of) Theorem 3 of [10].

We now actually prove a stronger result than Theorem 1:

**Theorem 1'.** Let \( G \) and \( \mu \) satisfy the hypotheses of Theorem 1. Let \( m \geq 2 \) and \( n \geq 1 \). Then there exists \( \varepsilon > 0 \) such that for each \( m \geq 1 \) there exist sets \( A_1, \ldots, A_m \subseteq \Gamma \), each having cardinality \( n \), such that
\[
A_1 A_2 \ldots A_m \subseteq E(\mu, \varepsilon).
\]

The method of [10] shows that \( A_1, A_2, \ldots, A_m \) may be chosen so that \( A_1 A_2 \ldots A_m \) has cardinality \( n^m \).

For the proof of the theorem, we use the generalized character notions of Šreider [11]. The reader might consult Taylor [12] for many results about generalized characters. Finally, let us point out again that we write \( \Gamma \) as a multiplicative group in the proof below.

Fix \( m \geq 1 \). We have several steps.

(A) We may assume that \( \Gamma \) is discrete, since \( \hat{\mu}(\gamma) \neq 0 \) and \( \Gamma \) not discrete implies that, for some neighborhoods (infinite sets in particular) \( U, V \) of 0 contained in \( \Gamma \),
\[
E(\mu, \varepsilon) \supseteq U \cdot V \cdot \gamma \quad \text{for some} \quad \varepsilon > 0.
\]
(B) We may assume that $G$ is metrizable. Indeed, let $\{\gamma_j\}_{1}^{\infty}$ be an infinite subset of $I$ such that $\limsup |\hat{\mu}(\gamma_j)| \neq 0$. Let $\Lambda$ be the subgroup of $I$ generated by $\{\gamma_j\}_{1}^{\infty}$.

It will be sufficient to find our sets $A_1, \ldots, A_m \subseteq \Lambda$. A moment's thought will convince the reader that we may therefore assume $I = \Lambda$.

(C) Let $\{\gamma_j\}_{1}^{\infty}$ be as in (B). We may assume ($G$ being metrizable) that $\lim \gamma_j = f$ exists in $L^{\infty}(\mu)$, the limit being weak-* of course. Then $f \neq 0$ a.e. $d\mu$, since

$$\int f d\mu = \lim \int \gamma_j d\mu \neq 0.$$ 

Therefore, $f^m$ (pointwise product in $L^{\infty}(\mu)$) is not identically zero a.e. $d\mu$, so the measure defined by $g \to \int gf^m d\mu \ (g \in C(G))$ is not identically zero. Therefore, there exists $\gamma_0 \in I$ such that

$$\int f^m \gamma_0 d\mu \neq 0.$$ 

We define $\varepsilon$ by

$$\varepsilon = \frac{1}{2^l} \left| \int f^m \gamma_0 d\mu \right| .$$

(D) Fix $n \geq 1$. Since $f$ is the weak-* limit of $\{\gamma_j\}$, there exist distinct $\varepsilon_{1,1}, \ldots, \varepsilon_{1,n} \in \{\gamma_j\}_{1}^{\infty}$ such that

$$\left| \int \varepsilon_{1,j} f^{m-1} \gamma_0 d\mu \right| > \left( \frac{1}{2} \right)^l \left| \int f^m \gamma_0 d\mu \right| \quad (1 \leq j \leq n).$$

(E) We now induct. Suppose that distinct $\varepsilon_{l,1}, \ldots, \varepsilon_{l,n} \in \{\gamma_j\}_{1}^{\infty}$ have been found for $l = 1, 2, \ldots, k-1 < m$ such that

$$\left| \int \varepsilon_{l,j_1} \cdots \varepsilon_{l,j_l} f^{m-l} \gamma_0 d\mu \right| > \left( \frac{1}{2} \right)^l \left| \int f^m \gamma_0 d\mu \right| ,$$

where $1 \leq j(1), \ldots, j(l) \leq n$, and $1 \leq l \leq k-1$.

We now find $\varepsilon_{k,1}, \ldots, \varepsilon_{k,n}$. Since $\lim \gamma_f = f$ weak-* in $L^{\infty}(\mu)$, there exist distinct $\varepsilon_{k,1}, \ldots, \varepsilon_{k,n} \in \{\gamma_j\}_{1}^{\infty}$ such that

$$\left| \int \varepsilon_{k,j_1} \cdots \varepsilon_{k,j_k} f^{m-k} \gamma_0 d\mu \right|$$

$$> \left( \frac{1}{2} \right)^{l+m} \left| \int \varepsilon_{k-1,j_1} \cdots \varepsilon_{k-1,j_{k-1}} f^{m-k+1} \gamma_0 d\mu \right| .$$

Now (4) (with $l = k-1$) and (5) yield (4) with $l = k$. This completes the induction, i.e., (4) now holds for $1 \leq j(1), \ldots, j(m) \leq n$ (and $m = l$).
By the choice (formula (2)) of \( \varepsilon \), we see that (4) (for \( l = m \)) implies

\[
\left| \int \varepsilon_{i,j(1)} \ldots \varepsilon_{m,j(m)} \gamma_0 \, d\mu \right| > \varepsilon \quad (1 \leq j(1), \ldots, j(m) \leq n).
\]

Setting

\[ A_i = \{ \varepsilon_{i,1}, \ldots, \varepsilon_{i,n} \} \quad (1 \leq i < m) \quad \text{and} \quad A_m = \{ \varepsilon_{m,1} \gamma_0, \ldots, \varepsilon_{m,n} \gamma_0 \}, \]

we see that \( A_1 A_2 \ldots A_m \leq B(\mu, \varepsilon) \) as required. This completes the proof of Theorem 1'.

3. Remarks.

(i) It is easy to see, using Riesz products, for example, that there is no necessary relationship between \( m, \varepsilon \) and \( \limsup |\mu(\gamma)| \).

(ii) That the \( A_i \) can be chosen so that \( A_1 A_2 \ldots A_m \) has cardinality \( n^m \) as simply done in our situation. Indeed, assume that \( A_1 A_2 \ldots A_{k-1} \) has cardinality \( n^{k-1} \) (1 < \( k \leq m \)). Since \( \lim \gamma_j = f \) weak-* , there is a \( J \) such that \( j \geq J \) implies (5) whenever \( \varepsilon_{k,(j)} = \gamma_j \) (and \( A_i = \{ \varepsilon_{i,1}, \ldots, \varepsilon_{i,n} \} \) for 1 < \( i < k \)). We choose \( \varepsilon_{k,1} = \gamma_j \), \( j \geq J \) arbitrarily. Then there must exist, in the infinite set \( \{ \gamma_j: i > j \} \), a \( \varepsilon_{k,2} = \gamma_i \) such that

\[ [A_1 A_2 \ldots A_{k-1}(\varepsilon_{k,1})] \cap A_1 A_2 \ldots A_{k-1}(\varepsilon_{k,2}) = \emptyset. \]

A simple induction now shows that \( A_1 A_2 \ldots A_m \) may be chosen with cardinality \( n^m \).

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REFERENCES


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