Harmonic functions in four variables with rational and algebraic $p_4$-associates

by R. P. GILBERT (Maryland)

Abstract. Integral representations for harmonic functions in four variables are investigated by means of an operator bearing close resemblance to the Whittaker-Bergman operator. The cases where the $p_4$-associate is algebraic is considered by means of the theory of double integrals on algebraic three-folds. When the $p_4$-associate is rational one obtains particulary interesting representations by considering the connections with Weierstrass integrals of the first, second, and third kinds defined over a Riemann surface. In addition, a residue theorem is given for a class of harmonic vectors $U = (u_1, u_2, u_3, u_4)$ satisfying the relations:

$$\epsilon_{mnrs} \frac{\partial u_r}{\partial x_s} = 0, \quad \frac{\partial u_r}{\partial x_r} = 0,$$

which are analogous to the vanishing of the curl and divergence in three-dimensions.

I. Introduction. The solutions of Laplace’s equation in four variables,

$$(1) \Box u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0,$$

may be generated by means of an integral operator $p_4[f]$, which bears a close resemblance to the Whittaker-Bergman operator $p_3[f]$ ([1], [2], [3], [4], [7], [8], [10], [11], [12], [13], [17]). The operator $p_4[f]$ transforms analytic functions of three complex variables into harmonic functions of four variables ([8], [12], [9]),

$$(2) u(X) = p_4[f] = \frac{1}{4\pi^3} \int_\mathcal{D} f(\tau, \eta, \xi) \frac{d\eta}{\eta} \frac{d\xi}{\xi},$$

where

$$\tau = x_1 \left(1 + \frac{1}{\eta \xi}\right) + ix_2 \left(1 - \frac{1}{\eta \xi}\right) + x_3 \left(\frac{1}{\xi} - \frac{1}{\eta}\right) + ix_4 \left(\frac{1}{\xi} + \frac{1}{\eta}\right),$$

$$\|X - X^0\| < \varepsilon, \quad X = (x_1, x_2, x_3, x_4), \quad X^0 = (x_1^0, x_2^0, x_3^0, x_4^0),$$
\( \mathcal{D} = C \times \Gamma \) is the product of a contour \( C \) in the \( \xi \)-plane, and a contour \( \Gamma \) in the \( \eta \)-plane, and \( \varepsilon > 0 \) is taken sufficiently small. We restrict \( \mathcal{D} \) further by insisting for each choice of \( f(\tau, \eta, \xi) \) the integrand is absolutely integrable ([3], [15]); here the double integral may be regarded as an iterated integral, and the orders of integration may be interchanged.

In order to realize how the operator \( p_4 \) transforms analytic functions in harmonic functions we may introduce the homogeneous, harmonic polynomials of degree \( n \) which are defined as follows ([6], [8], [9], [12]):

\[
\tau^n = \left\{ x_1 \left( 1 + \frac{1}{\eta \xi} \right) + ix_2 \left( 1 - \frac{1}{\eta \xi} \right) + x_3 \left( \frac{1}{\xi} - \frac{1}{\eta} \right) + ix_4 \left( \frac{1}{\xi} + \frac{1}{\eta} \right) \right\}^n
\]

\[
= \sum_{k,l=0}^{n} H_{n}^{k,l}(X) \xi^{-k} \eta^{-l}.
\]

In view of (3), these polynomials have an integral representation

\[
H_{n}^{k,l}(X) = -\frac{1}{4\pi^2} \int_{|\xi|=1} \frac{d\xi}{\xi} \int_{|\eta|=1} \frac{d\eta}{\eta} \tau^n \eta^k \xi^l,
\]

where \( k, l \) are integers from 0 to \( n \). Because of this representation, and because the \( H_{n}^{k,l}(X) \) form a complete set of harmonic polynomials, we may generate any harmonic function regular in the small of the origin,

\[
u(X) = \sum_{n=0}^{\infty} \sum_{k,l=0}^{n} a_{nkl} H_{n}^{k,l}(X),
\]

from an analytic function,

\[
f(\tau, \eta, \xi) = \sum_{n=0}^{\infty} \sum_{k,l=0}^{n} a_{nkl} \tau^n \eta^k \xi^l,
\]

by means of the operator \( p_4[f] \). We shall (following Kreyszig [12] and Bergman [4]) refer to (6) as the normalized associated function of \( \nu(X) \) with respect to \( p_4 \), or more concisely as the \( p_4 \)-associate of \( \nu(X) \).

II. Algebraic associates. In this section we consider the case, where

\[
\eta^{-1} \xi^{-1} f(\tau, \eta, \xi) = p_4(S; \tau, \eta, \xi) \left( \frac{p_4(S; \tau, \eta, \xi)}{p_4(S; \tau, \eta, \xi)} \right) = P_4(X; S; \eta, \xi)
\]

is a rational function of \( S, \tau, \eta, \xi \), and where \( S, \tau, \eta, \xi \) are connected by an algebraic equation (1)

\[
Q(S; \tau, \eta, \xi) = A_0(\tau, \eta, \xi) S^n + A_1(\tau, \eta, \xi) S^{n-1} + \ldots + A_n(\tau, \eta, \xi) = 0,
\]

(1) This case is an extension to four variables of the case considered by Professor Bergman in [5].
where the \( A_r(\tau, \eta, \xi) \) are polynomials in \( \tau, \eta, \xi \). If we multiply (8) by a suitable power of \( \eta \) and \( \xi \), we obtain

\[
(9) \quad \Omega(X; S; \eta, \xi) = A_0(X; \eta, \xi) S^n + A_1(X; \eta, \xi) S^{n-1} + \ldots + A_n(X; \eta, \xi) = 0,
\]

where the \( A_r(X; \eta, \xi) \) are polynomials in \( \eta, \xi \), and \( x_1, x_2, x_3, x_4 \).

The equation \( \Omega(X; S; \eta, \xi) = 0 \) defines for each fixed value of \( X \) an algebraic surface in \( S, \eta, \xi \). The surface may also be represented parametrically as

\[
(10) \quad S = \chi(X; \alpha, \beta), \quad \eta = \varphi(X; \alpha, \beta), \quad \xi = \psi(X; \alpha, \beta),
\]

where \( \chi, \varphi, \psi \) are algebraic functions in \( \alpha, \beta \). Using this representation we consider the sets of equations

\[
\chi(X; \alpha, \beta) = \chi(X; \alpha', \beta'),
\]

\[
\varphi(X; \alpha, \beta) = \varphi(X; \alpha', \beta'),
\]

\[
\psi(X; \alpha, \beta) = \psi(X; \alpha', \beta')
\]

and

\[
\chi(X; \alpha, \beta) = \chi(X; \alpha', \beta') = \chi(X; \alpha'', \beta''),
\]

\[
\varphi(X; \alpha, \beta) = \varphi(X; \alpha', \beta') = \varphi(X; \alpha'', \beta''),
\]

\[
\psi(X; \alpha, \beta) = \psi(X; \alpha', \beta') = \psi(X; \alpha'', \beta'').
\]

There is an \( \infty^3 \) of solutions to (11) and this will be in general a line through which pass two nappes of the surface. This line is called a double curve. There will be only a limited number of solutions to (12); these are the triple points of the surface, through which passes three double lines and three nappes of the surface. The double curves, and triple points are singularities of the surface. If by a birational transformation the surface is mapped into another algebraic surface, whose only singularities are a double curve and its triple points, the singularities are called ordinary singularities ([16]).

We now suppose for \( X = X^0 \) (an initial point) the algebraic surface \( \Omega(X^0; S; \eta, \xi) = 0 \) has just ordinary singularities. We consider integrals on \( \Omega = 0 \) of the form,

\[
(13) \quad U(X) = -\frac{1}{4\pi^2} \int \int \frac{P_1(X; S; \eta, \xi)}{P_2(X; S; \eta, \xi)} d\eta d\xi,
\]

where \( X \in N(X^0) \), and \( N(X^0) \) is a sufficiently small neighborhood of \( X^0 \) such that \( \Omega(S; X^0; \eta, \xi) = 0 \) has only ordinary singularities. It is convenient sometimes to write

\[
(14) \quad \frac{P_1(X; S; \eta, \xi)}{P_2(X; S; \eta, \xi)} = \frac{R(X; S; \eta, \xi)}{\partial \Omega(X; S; \eta, \xi)/\partial S},
\]
where $R(X; S; \eta, \xi)$ is rational in $S, \eta, \xi$. In Picard and Simart ([16]) it is stated that the necessary and sufficient conditions for an integrand (14) to always correspond to a finite valued integral (13) is that $R(X; S; \eta, \xi) = Q(X; S; \eta, \xi)$ be a polynomial of degree $n - 4$, and that $Q(X; S; \eta, \xi)$ pass through the double curve of $\mathcal{Q} = 0$. In this case, we call the integral (13) a double integral of the first kind ([16]). There is a number $p_{\phi}(X)$ associated with the surface $\mathcal{Q} = 0$, called the geometric genus or Flachengeschlecht, which plays a role similar to the Riemannian genus of an algebraic curve. This number $p_{\phi}(X)$ is the number of linearly independent double integrals of the first kind, that is the number of linearly independent polynomials of degree $n - 4$, $Q_k(X; S; \eta, \xi)$, which pass through the double curve of $\mathcal{Q} = 0$. It is understood that $p_{\phi} = p_{\phi}(X)$ is essentially independent of $X$ (1). If $B$ is the set of points for which the surface $\mathcal{Q} = 0$ only has ordinary singularities, the geometric genus is given by the simple formula ([16])

\[
p_{\phi} = \frac{(n-1)(n-2)(n-3)}{6}.
\]

This follows from the fact that $p_{\phi}$ is invariant under a birational transformation, and that $p_{\phi}$ is the number of arbitrary constants in the most general surface of degree $n - 4$ with no singular points. Consequently, the most general integrals of the first kind representing $u(X)$ may be expressed as

\[
u(X) = -\frac{1}{4\pi^2} \sum_{k=1}^{p_{\phi}} \int_D \int A_k \frac{Q_k(X; S; \eta, \xi)}{\partial \mathcal{Q}(X; S; \eta, \xi)/\partial S} d\eta d\xi.
\]

We say that the integral (13) is a double integral of the second kind over the algebraic surface ([5], [9]), if

\[
\int_D \int \frac{P_1(X; S; \eta, \xi)}{P_2(X; S; \eta, \xi)} d\eta d\xi - \int_D \int \left( \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} \right) d\eta d\xi
\]

remains finite for all $D$. ($U$ and $V$ are rational functions.) Furthermore, if $\mathcal{Q} = 0$ is an algebraic surface with just ordinary singularities it is known that the integrals of the second kind representing $u(X)$ have the form

\[
u(X) = -\frac{1}{4\pi^2} \int_D \int \frac{P(X; S; \eta, \xi)}{\partial \mathcal{Q}(X; S; \eta, \xi)/\partial S} d\eta d\xi + \frac{1}{4\pi^2} \int_D \int \left( \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} \right) d\eta d\xi,
\]

where $P(X; S; \eta, \xi)$ is a polynomial which vanishes on the double curve of $\mathcal{Q} = 0$. This result suggests the definition ([15]) that a set of integrals

\footnote{See footnote (1), p. 335.}
of the second kind are distinct if any linear combination is not equal to the form
\[
\int \int \left( \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} \right) d\eta d\xi.
\]

From this definition follows the fundamental theorem on double integrals of the second kind ([15]):

For an algebraic surface \( \mathcal{Q} = 0 \) there are \( \varrho_0 \) distinct double integrals of the second kind, \( I_1, I_2, \ldots, I_{\varrho_0} \), such that any double integral of the second kind may be written in the form
\[
\alpha_1 I_1 + \alpha_2 I_2 + \ldots + \alpha_{\varrho_0} I_{\varrho_0} + \int \int \left( \frac{\partial U}{\partial \eta} + \frac{\partial V}{\partial \xi} \right) d\eta d\xi,
\]
where the \( \alpha_k \) are constants and \( U, V \) are rational in \( S, \eta, \xi \). Furthermore, if \( \mathcal{Q} = 0 \) is simply connected the number \( \varrho_0 \) is given by the formula
\[
\varrho_0 = N - 4p - (n-1) - (p-1),
\]
where \( N \) is the class of the surface, \( p \) is the genus of an arbitrary plane section, and \( n \) is the degree. \( (p-1) \) is the number of particular irreducible algebraic curves \( \{C_i\}_{i=1}^{p-1} \), which may be drawn on \( \mathcal{Q} = 0 \), such that there does not exist a total differential of the third kind, having for its "logarithmic curves" one or more of the curves \( \{(C_i)_{i=1}^{p-1} + \text{the curve at } \infty \} \), but if another arbitrary curve \( C_q \) is added to the set there exists an integral having one or more curves from this new set as its "logarithmic curves".

The class \( N \) of a surface defined by the equation \( \mathcal{Q}(X; S, \eta, \xi) = 0 \) is the number of values of \( \xi = \xi_i, \ i = 1, 2, \ldots, N \), for which the genus of the Riemann surface, \( E \{ \mathcal{Q} = 0 \} \cap \{ \xi = \xi_i \} \), is less than \( p \). It is clear, that the genus will be diminished by one when the plane \( \xi = \xi_i \) becomes tangent to the surface \( \mathcal{Q} = 0 \) at a simple point; such a point constitutes in general a double point of the surface. From the above discussion we realize that if the integral representing \( u(X) \) is of the second kind one may write
\[
(18) \quad u(X) = \sum_{k=1}^{\infty} \alpha_k I_k(X) + \frac{1}{4\pi^2} \int \int \left( \frac{\partial U(X; S, \eta, \xi)}{\partial \eta} + \frac{\partial V(X; S, \eta, \xi)}{\partial \xi} \right) d\eta d\xi,
\]
where the \( \{I_k(X)\} \) are a unique linearly independent set of integrals of the second kind over \( \mathcal{Q} = 0 \).

**III. Rational associates.** When \( f(\tau, \eta, \xi) \) is a rational function of \( \tau, \eta, \xi \) it is possible to obtain representation formulae for \( u(X) \) in terms of certain classical functions. In certain special cases we shall be able to reproduce the representations obtained by Bergman ([1], [3], [17]).
and Mitchell (13) for harmonic functions of three variables with algebraic associates. For instance let us suppose

\begin{equation}
(19) \quad f(\tau, \eta, \xi) = \frac{q_1(\tau, \eta, \xi)}{q_2(\tau, \eta, \xi)} \eta \xi = \frac{Q_1(X; \eta, \xi)}{Q_2(X; \eta, \xi)}
\end{equation}

and let us consider

\[ u(X) = -\frac{1}{4\pi^2} \int \int_{\mathcal{D}} \frac{Q_1(X; \eta, \xi)}{Q_2(X; \eta, \xi)} d\eta d\xi, \quad \mathcal{D} = \Gamma \circ C. \]

The singularity manifold of the integrand may be represented as

\begin{equation}
(20) \quad \{ Z^* = E \{ Q_2(X; \eta, \xi) = 0 \}, \quad \text{or as} \quad \{ Z^* = E \{ \xi = A_\nu(X; \eta) ; \nu = 1, 2, \ldots, m \}. \}
\end{equation}

In general for each fixed \( \eta^0 \in \Gamma \), there will be \( \mu \) roots, \( \xi_{k_1}, \xi_{k_2}, \ldots, \xi_{k_\mu} \), inside \( C \) and \( m - \mu \) roots, \( \xi_{k_{m+1}}, \ldots, \xi_{k_m} \), outside \( C \). As \( \eta \) varies over \( \Gamma \) the roots \( \xi_\nu(\eta) \) move in the \( \xi \)-plane, and some may cross over \( C \). If we restrict \( X \) such that,

\begin{equation}
(21) \quad X \notin M^3 \equiv E \left\{ \left| \prod_{0 \leq k \leq j \leq m} [A_j(X; \eta) - A_k(X; \eta)] \right| = 0 ; \eta \in \Gamma \right\},
\end{equation}

then there cannot be more than a first order pole of the integrand on the domain of integration, and in this case the integrand is absolutely integrable. One then has

\begin{equation}
(22) \quad u(X) = -\frac{1}{4\pi^2} \int \int_{\mathcal{D}} \frac{Q_1(X; \eta, \xi)}{Q_2(X; \eta, \xi)} d\eta d\xi = \frac{1}{2\pi i} \sum_{\mu=1}^{m} \int_{\Gamma_\mu} \frac{Q_1(X; \eta, \xi_\mu)}{\partial Q_2(X; \eta, \xi_\mu)} d\eta,
\end{equation}

where \( \Gamma_\mu \) is that subset of the path \( \Gamma \) for which \( \xi_\mu = A_\mu(X; \eta) \) lies inside of \( C \). Consequently, one may express each residue of the integrand (19) as an Abelian integral

\begin{equation}
(23) \quad u(X) = \frac{1}{2\pi i} \int_{\Gamma^*} \frac{Q_1(X; \eta, \xi)}{\partial Q_2(X; \eta, \xi)} d\eta,
\end{equation}

where \( \Gamma^* = \sum_{\mu=1}^{m} \Gamma_\mu \) is taken over the Riemann surface, \( \mathcal{R}(X) \), defined by \( Q_2(X; \eta, \xi) = 0 \).

Before proceeding further, we should like to make some remarks about the topology of \( \mathcal{R}(X) \). In general, the genus \( g(X) \) of \( \mathcal{R}(X) \) will be constant, and \( \mathcal{R}(X) \) will have \( m \)-sheets over the \( \eta \)-plane. However, we note as Bergman has done in the case of three variables that there will be certain exceptional points. To locate these exceptional points we express

\begin{equation}
(24) \quad Q_2(X; \eta, \xi) = \sum_{r=0}^{m} q_r(X; \eta) \xi^r,
\end{equation}
where the \( q_n(X; \eta) \) are polynomials in \( X, \eta \). We designate the following sets:
\[
\mathcal{S}_2 = \mathbb{E} \{ X \mid q_n(X; \eta) = 0 \}, \\
\mathcal{G}_2 = \mathbb{E} \{ X \mid A(X; \eta) \equiv 0 \}, \\
\mathcal{G}_3 = \left[ \mathbb{E} \{ X \mid A(X; \eta) = 0 \} \cap \mathbb{E} \left\{ X \left| \frac{\partial A}{\partial \eta} = 0 \right. \right\} \right] \cup \\
\mathbb{E} \{ X \mid q_n(X; \eta) = 0 \} \cap \mathbb{E} \left\{ X \left| \frac{\partial q_n}{\partial \eta} = 0 \right. \right\},
\]
and
\[
\mathcal{G}_2 = \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4. 
\]
As Mitchell has pointed out ([13]) when \( X \in \mathcal{G}_2, \mathcal{R}(X) \) has less than \( n \)-sheets; when \( X \in \mathcal{G}_3, q_3(X; \eta, \xi) \) has a repeated irreducible factor involving \( \xi \); if \( X \in \mathcal{G}_4, A(X; \eta) \) or \( q_4(X; \eta) \) has a multiple root and two branch points coincide.

Now, as Bergman ([1], [3], [17]) has done in the case of algebraic \( \mathcal{p}_n \)-associates, we make use of the Weierstrass decomposition formula ([18], p. 264) for algebraic functions defined over a Riemann surface

\[
F(X; \eta, \xi) = \sum_{s=1}^r c_s(X) H_s(X; \eta, \xi) - \sum_{s=1}^r \left[ g_s^*(X) H_s(X; \eta, \xi) - g_4(X) H_s^*(X; \eta, \xi) \right] + \frac{d}{d \eta} \left[ \sum_{s=1}^r F_s(X; \eta, \xi) \right],
\]
\[
\sum_{s=1}^r c_s(X) = 0,
\]
where \( H_s(X; \eta, \xi), H_s^*(X; \eta, \xi), H(X; \eta, \xi), H_s(X; \eta, \xi), H_s^*(X; \eta, \xi) \) are Weierstrass integrands of the first, second, and third kind respectively. The \( F_s(X; \eta, \xi) \) are rational functions of \( \eta, \xi, p \) is the genus, \( r = r(X) \) is the number of infinity points of \( F(X; \eta, \xi) \), and \( c_s(X), g_s(X), g_s^*(X) \), are algebraic functions of \( X \). The number of infinity points of \( F = Q_1/Q_2 \) is the same as the number of zeros of the coefficient \( q_4(X; \eta) \).

In the situations discussed by Bergman ([1], [3], [17]) and Mitchell ([13]) for integrals of the type (23), \( I^* \) was always a closed curve over \( \mathcal{R}(X) \). With our case \( I^* \) is in general a sum of \( m \) disjoint segments on \( \mathcal{R}(X) \). The special case where \( I^* \) becomes a closed curve corresponds to when each of the roots \( \xi = A_\mu(X; \eta) \) (\( \mu = 1, 2, \ldots, m \), remains in-

(*) The superscripts on the sets \( \mathcal{G}_i \) indicate that the sets are of real dimension two. This convention for superscripts on sets will be used throughout this paper.
side $C$ for all $\eta \in \Gamma$. In that case we obtain representations identical to those of Bergman ([1], [3], [17]) and Mitchell ([13]).

In order to evaluate (23) we must introduce certain integrals on $\Re(X)$ associated with the integrands of the first, second, and third kind:

$$\int_C H_a(X; \eta, \xi) \, d\eta, \quad \int_C H^*_a(X; \eta, \xi) \, d\eta \quad (a = 1, 2, \ldots, p),$$

$$\int_C H(X; \eta_r, \xi_r; \eta, \xi) \, d\eta \quad (r = 1, 2, \ldots, r),$$

and their periods taken over the $p$ cycles $K_p(X)$, and $p$ conjugate cycles $K^*_p(X)$

$$2 \omega_{ap}(X), \quad 2 \eta_{ap}(X), \quad \Omega_p(X; \eta_r, \xi_r) \quad (a = 1, 2, \ldots, p),$$

$$2 \omega^*_{ap}(X), \quad 2 \eta^*_{ap}(X), \quad \Omega^*_p(X; \eta_r, \xi_r) \quad (\beta = 1, 2, \ldots, p).$$

In addition we introduce integrals taken over open paths on $\Re(X)$:

$$J_a(X; \eta, \xi) = \int_{(a_0, b_0)} H_a(X; \eta', \xi') \, d\eta',$$

$$J^*_a(X; \eta, \xi) = \int_{(a_0, b_0)} H^*_a(X; \eta', \xi') \, d\eta' \quad (a = 1, 2, \ldots, p)$$

and

$$J(X; \eta, \xi; \eta_r, \xi_r; \eta_0, \xi_0)$$

$$= \int_{(a_0, b_0)} \{ H(X; \eta_r, \xi_r; \eta', \xi') - H(X; \eta_0, \xi_0; \eta', \xi') \} \, d\eta'.$$

The integral (30) has the following primitive period system:

$$2\pi i, \quad \log \frac{E^*_p(X; \eta_r, \xi_r)}{E^*_p(X; \eta_0, \xi_0)}, \quad \log \frac{E_p(X; \eta_r, \xi_r)}{E_p(X; \eta_0, \xi_0)} \quad (\beta = 1, 2, \ldots, p),$$

where the functions $E_p, E^*_p$ are defined as

$$E_p(X; \eta, \xi) = \exp \{ \Omega_p(X; \eta, \xi) \},$$

$$E^*_p(X; \eta, \xi) = \exp \{ \Omega^*_p(X; \eta, \xi) \}.$$

Lastly, we shall need the integrals of the third kind

$$\Omega(X; \eta, \xi; \eta_r, \xi_r; \eta_0, \xi_0) = \int_{(a_0, \xi_0)} \{ H(X; \eta, \xi; \eta', \xi') \} \, d\eta',$$

and the functions

$$E(X; \eta, \xi; \eta_r, \xi_r; \eta_0, \xi_0) = \exp \{ \Omega(X; \eta, \xi; \eta_r, \xi_r; \eta_0, \xi_0) \}.$$
According to Weierstrass ([18], pp. 373, 374, 398) certain relations exist between the integrands $H_{\beta}(X; \eta, \xi)$, $H^*_{\beta}(X; \eta, \xi)$, $H(X; \eta_0, \xi_0; \eta, \xi)$ and the $E$-functions,

$$H_{\beta}(X; \eta, \xi) = \frac{d}{d\eta} J_{\beta}(X; \eta, \xi)$$

$$= \frac{1}{2\pi i} \sum_{\alpha=1}^{p} \left\{ \omega_{\beta\alpha} \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) - \omega_{\beta\alpha}^* \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) \right\},$$

(35)

$$H^*_{\beta}(X; \eta, \xi) = \frac{d}{d\eta} J^*_{\beta}(X; \eta, \xi)$$

$$= \frac{1}{2\pi i} \sum_{\alpha=1}^{p} \left\{ \eta_{\beta\alpha} \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) - \eta_{\beta\alpha}^* \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) \right\},$$

and

(36) $H(X; \eta_0, \xi_0; \eta, \xi) - H(X; \eta_0, \xi_0; \eta, \xi) = \frac{d}{d\eta} \log E(X; \eta, \xi; \eta_0, \xi_0; \eta, \xi)$

$$- \frac{1}{2\pi i} \sum_{\alpha=1}^{p} \left\{ \log \frac{E_{\alpha}(X; \eta_0, \xi_0)}{E_{\alpha}(X; \eta, \xi)} \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) - \frac{d}{d\eta} \log \frac{E_{\alpha}(X; \eta_0, \xi_0)}{E_{\alpha}(X; \eta, \xi)} \right\}.$$

The function $F(X; \eta, \xi)$ may now be expressed in terms of derivatives of the $E$-functions by subtracting from it the sum

$$\sum_{r=1}^{r} c_r H(X; \eta_r, \xi_r; \eta, \xi),$$

(37) $F(X; \eta, \xi) - \sum_{r=1}^{r} c_r H(X; \eta_r, \xi_r; \eta, \xi)$

$$= \sum_{r=1}^{r} c_r \frac{d}{d\eta} \log E(X; \eta, \xi; \eta, \xi; \eta, \xi) +$$

$$+ \sum_{\alpha=1}^{p} \left\{ C_{\alpha}(X) \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) - C_{\alpha}(X) \frac{d}{d\eta} \log E_{\alpha}(X; \eta, \xi) \right\} +$$

$$+ \frac{d}{d\eta} \left[ \sum_{r=1}^{r} F_r(X; \eta, \xi) \right],$$
where
\[ C^*_a(X) = \frac{1}{2\pi i} \left( \sum_{r=1}^{\mu} c_r \log \frac{E^*_a(X; \eta_r, \xi_r)}{E^*_a(X; \eta_0, \xi_0)} - 2 \sum_{\beta=1}^{p} [\omega^*_a(X) g^*_\beta(X) - \eta^*_a(X) g^*_\beta(X)] \right), \]
(38)
\[ C_a(X) = \frac{1}{2\pi i} \left( \sum_{r=1}^{\mu} c_r \log \frac{E_a(X; \eta_r, \xi_r)}{E_a(X; \eta_0, \xi_0)} - 2 \sum_{\beta=1}^{p} [\omega_a(X) g^*_\beta(X) - \eta_a(X) g^*_\beta(X)] \right). \]

We return to the evaluation of (23), and recall that \( \Gamma_\mu \) is that subset of \( \Gamma \) for which the root \( \xi_\mu = A_\mu(X; \eta) \) lies inside \( C \). The root \( \xi_\mu \) crosses \( C \) a finite number of times, providing \( C \) is sufficiently smooth, since \( f(\tau, \eta, \xi) \) is a rational function. Furthermore, we assume that \( \xi_\mu \) lies inside \( C \) for \( \eta \) contained in the union of intervals,
\[ \Gamma_\mu = \bigcup_{j=1}^{k_\mu} (\eta^{2j-1}_\mu, \eta^j_\mu), \quad \eta^l_\mu = \eta^l_\mu(X) \in \Gamma. \]
(39)

Consequently, one has the result:

**Theorem 1.** If the \( p_4 \)-associate of \( u(X) \) is a rational function of \( \tau, \eta, \xi \) then whenever \( X \in \mathbb{S}^3 \), \( u(X) \) may be represented as
\[ u(X) = \sum_{\mu=1}^{m} \sum_{j=1}^{k_\mu} \left\{ \sum_{r=1}^{\mu} c_r(X) \log \frac{E(X; \eta^{2j}_\mu, \xi^{2j}_\mu; \eta_r, \xi_r)}{E(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu; \eta_r, \xi_r; \eta_0, \xi_0)} + \right. \\
+ \sum_{a=1}^{p} \left[ \frac{C^*_a(X) \log \frac{E_a(X; \eta^{2j}_\mu, \xi^{2j}_\mu)}{E_a(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu)} - C_a(X) \log \frac{E^*_a(X; \eta^{2j}_\mu, \xi^{2j}_\mu)}{E^*_a(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu)} \right] + \\
\left. + \sum_{r=1}^{\mu} [F_r(X; \eta^{2j}_\mu, \xi^{2j}_\mu) - F_r(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu)] \right\}. \]
(40)

**Theorem 2.** If in Theorem 1 the roots \( \xi_\mu(X; \eta) \) all lie inside for all \( \eta \in \Gamma \), then the representation (40) reduces to the form
\[ u(X) = \sum_{\beta=1}^{p} \left\{ \sum_{r=1}^{\mu} \left[ c'_\beta(X) \Omega^*_\beta(X; \eta_r, \xi_r) + c'_\beta(X) \Omega^*_\beta(X; \eta_r, \xi_r) \right] - \right. \\
- \sum_{a=1}^{p} \left[ g^*_a(X) \omega_a(X) - g^*_a(X) \omega_a(X) - g_a(X) \eta_a(X) - g_a(X) \eta_a(X) \right] + \\
\left. + \text{Res} \sum_{r=1}^{\mu} F_r(X; \eta, \xi) \right\}. \]
(41)
Theorem 2 is essentially Bergman's result for three-dimensional harmonic functions, and the reader is referred to his paper [5] on this subject, and also to his book [4].

IV. A residue theorem for harmonic vectors. In this section we introduce the harmonic vector \( U(X) = (u_1, u_2, u_3, u_4) \), \( \Box u_k = 0 \), where the components are defined as follows:

\[
\begin{align*}
    u_k(X) & = -\frac{1}{4\pi^2} \int_{|\tau|=1} \int_{|\eta|=1} f(\tau, \eta, \xi) N_k(\eta, \xi) \frac{d\eta}{\eta} \frac{d\xi}{\xi}, \\
    N(\eta, \xi) & = \frac{1}{\eta \xi} (\eta^2 + 1, i[\eta \xi - 1], \eta - \xi, i[\eta + \xi]).
\end{align*}
\]

(42)

\( U(X) \) satisfies conditions similar to the vanishing of the divergence and curl, namely

\[
\frac{\partial u_r}{\partial x_r} = 0, \quad \epsilon_{mnr} \frac{\partial u_r}{\partial x_n} = 0,
\]

(43)

where repeated indices indicate summation and \( \epsilon_{mnr} \) is a permutation symbol.

We are interested in integrals of the form

\[
-4\pi^2 \int_{\mathbb{S}'} U(X) \cdot dX = \int_{\mathbb{S}'} dX \cdot \int_{\mathbb{S}} f(\tau, \eta, \xi) N(\eta, \xi) \frac{d\eta}{\eta} \frac{d\xi}{\xi},
\]

(44)

where \( \mathbb{S}' \) is a smooth oriented curve, such that \( \mathbb{S}' \cap \mathbb{S}^2 = 0 \), and \( f(\tau, \eta, \xi) \) is a rational function of \( \tau, \eta, \xi \). We shall assume that \( \mathbb{S}' \) intersects

\[
\mathcal{M}^3 = E\{X \mid \Lambda(X; \eta) = 0; \eta \in \Gamma\}
\]

and

\[
\mathcal{N}^3 = E\{X \mid q_n(X; \eta) = 0; \eta \in \Gamma\}
\]

in a discrete set of points \( \{X_k\}_{k=1}^{s} \), and that these points subdivide \( \mathbb{S} \) into open segments \( \mathbb{S}'_k = \mathbb{S}(X_{k-1}, X_k), \mathbb{S}' = \bigcup_{k=1}^{s} \overline{\mathbb{S}'_k} \). If \( X^0 \in \mathbb{S}'_k \), \( \Gamma \) does not meet any of the branch points of \( \Re(X) \), because these points are contained in the union of the sets,

\[
E\{\eta \mid \Lambda(X^0; \eta) = 0\} \cup E\{\eta \mid q_n(X^0; \eta) = 0\}.
\]

Furthermore, for \( X^0 \in \mathbb{S}'_k \) there are no poles of

\[
F(X^0; \eta, \xi) = \frac{Q_2(X^0; \eta, \xi)}{\partial Q_2(X^0; \eta, \xi) / \partial \xi} \quad \text{on} \quad Q_2(X^0; \eta, \xi) = 0,
\]

since the discriminant \( \Lambda(X^0; \eta) \) is computed by eliminating \( \xi \) between \( Q_2 = 0 \) and \( \partial Q_2 / \partial \xi = 0 \).
Consequently we consider the integral

\[ 4\pi^2 \sum_{k=1}^{s} \int_{\mathcal{S}_k} U(X) \circ dX = \sum_{k=1}^{s} \int_{\mathcal{S}_k} dX \circ \int_{\mathfrak{D}} \int_{\mathfrak{D}} f(\tau, \eta, \xi) N(\eta, \xi) \frac{d\eta}{\eta} \frac{d\xi}{\xi} \]

(45)

\[ = 2\pi i \sum_{k=1}^{s} \int_{\mathcal{S}_k} dX \circ \int_{\mathfrak{D}} Q_1(X; \eta, \xi) N(\eta, \xi) d\eta, \]

\[ (\Gamma^* = \sum_{\mu=1}^{m} \Gamma_{\mu}, Q^* = \eta \xi Q_2, \text{ and } N^* = \eta \xi N) \]

(46)

\[ = 2\pi i \sum_{k=1}^{s} \int_{\mathcal{S}_k} dX \circ \left\{ \sum_{\mu=1}^{m} \sum_{j=1}^{k_p(X)} C_{\mu}(X) \log E(X; \eta^{2j}_\mu, \xi^{2j}_\mu; \eta_\tau, \xi_\tau; \eta_0, \xi_0) + \right. \]

\[ + \sum_{a=1}^{p} C^*_a(X) \log \frac{E_a(X; \eta^{2j}_\mu, \xi^{2j}_\mu)}{E_a(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu)} - \sum_{a=1}^{p} C_a(X) \log \frac{E^*_a(X; \eta^{2j}_\mu, \xi^{2j}_\mu)}{E^*_a(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu)} + \]

\[ \left. + \sum_{\tau=1}^{r(X)} \left[ F_\tau(X; \eta^{2j}_\mu, \xi^{2j}_\mu) - F_\tau(X; \eta^{2j-1}_\mu, \xi^{2j-1}_\mu) \right] \right\}, \]

where \( c_\mu(X), C_\mu(X), C^*_a(X) \), etc. are vector functions whose components are terms similar to those of (38), arising from the Weierstrass decomposition of the components \( F_\tau = \frac{Q_1 N^*}{\partial Q_2 \partial \xi} \), on the Riemann surface defined by \( Q^*_2 = 0 \).

Because the integrand in (45) is absolutely integrable we may interchange the \( X \) and \((\eta, \xi)\) orders of integration. Then in order to facilitate evaluation it is convenient to make the transformation from the \( X \)-space to the \( \tau \)-plane by \( \tau = \tau(X; \eta, \xi) \), where \( \eta, \xi \) are held fixed. We are led to consider the integral

\[ I_\alpha = \int_{\mathfrak{D}} \int_{\mathfrak{D}} \frac{d\eta}{\eta} \frac{d\xi}{\xi} \int_{\mathfrak{D}(\eta, \xi)} \int_{\mathfrak{D}(\eta, \xi)} f(\tau, \eta, \xi) d\tau, \]

(47)

where \( \mathfrak{D}(\eta, \xi) \) is the image of \( \mathfrak{D} \) under the mapping \( X \to \tau \) for fixed \( \eta, \xi \). Since, \( q_\alpha(\tau, \eta, \xi) \) is a polynomial, we may decompose it as

\[ q_\alpha(\tau, \eta, \xi) = [\tau - \tau_1(\eta, \xi)]^m[\tau - \tau_2(\eta, \xi)]^n ... [\tau - \tau_k(\eta, \xi)]^k \]

(48)

for all \((\eta, \xi) \notin E \{ (\eta, \xi) | \prod_{0 \leq \mu, \nu < k} [\tau_\mu(\eta, \xi) - \tau_\nu(\eta, \xi)] = 0 \}, \) that is \( q_\alpha(\tau, \eta, \xi) \) will have the factorization (48) with the exception of those \((\eta, \xi)\) lying
on a certain set of $n(n + 1)/2$ analytic surfaces $\xi = \psi_k(\eta)$. Hence, we obtain,

$$I_2 = \int \int_{\mathbb{C}} \frac{d\eta}{\eta} \frac{d\xi}{\xi} \sum_{j=1}^{k} n(\mathcal{Z}; \tau_j) \cdot \frac{\partial^{j-1}}{\partial \tau^{j-1}} \left[ (\tau - \tau_j(\eta, \xi))^{j-1} \frac{q_1(\tau, \eta, \xi)}{q_2(\tau, \eta, \xi)} \right]_{\tau=\tau_j},$$

where $n(\mathcal{Z}; \tau_j)$ is the winding number, or index, of $\mathcal{Z}(\eta, \xi)$ with respect to $\tau_j$. It is clear, that for certain subsets in $(\eta, \xi) \in \mathbb{D}$, $n(\mathcal{Z}; \tau_j)$ will be zero, and for other subsets of $\mathbb{D}$ it will take on positive integer values. In the special case, where $\mathcal{Z}'$ is chosen such that $\mathcal{Z}(\eta, \xi)$ winds about $\tau_j$ at most once, and where $l_j \equiv 1$, we may write

$$I_2 = \int \int_{\mathbb{D}} \frac{d\eta}{\eta} \frac{d\xi}{\xi} \left\{ \sum_{j=1}^{k} n(\mathcal{Z}; \tau_j) \frac{q_1(\tau, \eta, \xi)}{\partial q_2(\tau, \eta, \xi)/\partial \tau} \right\}$$

where $\tau_j(\eta, \xi)$ is a root of $q_2(\tau, \eta, \xi) = \sum_{r=0}^{k} a_r(\eta, \xi) \tau^r = 0$. The index $n(\mathcal{Z}; \tau_j)$ may be thought of here as a set function, since when $(\eta, \xi) \in \mathbb{D}^*_j \subset \mathbb{D}$, $n(\mathcal{Z}; \tau_j) = 1$, whereas for $(\eta, \xi) \in \mathbb{D} - \mathbb{D}^*_j$, $n(\mathcal{Z}; \tau_j) = 0$. Now, for each fixed $\eta = \eta^0 \in \Gamma$, there are certain values of $\xi \in C$ for which $\tau_j$ lies inside $\mathcal{Z}(\eta, \xi)$. Let us suppose that the segments of $C$ for which this occurs may be expressed as

$$C_j^* = \bigcup_{i=1}^{l_j(\eta^0)} C_{ji}(\eta^0) \equiv \bigcup_{i=1}^{l_j(\eta^0)} C(\xi_{2i-1}^j(\eta^0), \xi_{2i}^j(\eta^0)), \tag{50}$$

where the $\xi_{2i-1}^j, \xi_{2i}^j$ are the end points of the segments on $C$. As $\eta^0$ varies along $\Gamma$ the intervals $C_{ji}(\eta^0)$ and their number vary. If for a particular $\eta = \eta'$ there is no subset of $C$ for which $\tau_j$ lies inside $\mathcal{Z}$ we shall set $C_j^* = 0$; consequently, we may write

$$I_2 = \sum_{j=1}^{k} q_1(\tau_j, \eta, \xi) \frac{d\xi}{\xi} \frac{d\eta}{\eta} = \sum_{j=1}^{k} \int_{\mathbb{D}} \frac{d\eta}{\eta} \int_{C_j^*} \frac{q_1(\tau_j, \eta, \xi)}{\partial q_2(\tau, \eta, \xi)/\partial \tau} \frac{d\xi}{\xi}. \tag{51}$$

If $q_2(\eta, \xi)$ is of degree $m$ and $q_1(\eta, \xi)$ is of degree $m - 4$, the integral $I_2$ may be represented as an integral of the first kind provided $q_1(\eta, \xi) = 0$ passes through the double lines of $q_2(\eta, \xi) = 0$. Here we may represent $q_2(\eta, \xi)$ as a linear combination of $p_{\mu} = \binom{m-1}{3}$ independent polynomials $Q_\nu(\eta, \xi)$ ($\nu = 1, 2, ..., p_{\mu}$) and consequently $q_1(\eta, \xi)$ may be represented in terms of the integrands of the first kind $H_{\mu}(\eta, \xi)$ $= Q_\mu \frac{\partial q_2}{\partial \eta}$. Consequently,

$$I_2 = \sum_{j=1}^{k} \sum_{\mu=1}^{p_{\mu}} a_{\mu} \int_{C_j^*} H_\mu(\eta, \xi) d\xi d\eta, \tag{52}$$

and we have the result:
THEOREM 3. Let \( U(X) = p_{r}[f/\mathbb{N}] \) be a harmonic vector defined as above with the rational associate \( f(\tau, \eta, \xi) = q_1(\tau, \eta, \xi)/q_2(\tau, \eta, \xi) \). Furthermore, let \( q_2(\tau, \eta, \xi) \) be of degree \( m \), \( q_1(s, \eta, \xi) \) be of degree \( m - 4 \), and let \( q_1 = 0 \) pass through the double line of \( q_2 = 0 \). Then the integral of \( U(X) \cdot dX \) taken about a smooth, oriented curve \( \mathcal{S}' \) may be represented as follows:

\[
\int_{\mathcal{S}'} U(X) \cdot dX = \sum_{\mu=1}^{\infty} \sum_{j=1}^{k_{\mu}} \int_{\mathcal{S}'} \left( \sum_{r=1}^{t} c_r(X) \log \frac{E(X; \eta_{\mu}^j, \xi_{\mu}^j; \eta_{r}, \xi_{r}); \eta_{0}, \xi_{0})}{E(X; \eta_{\mu}^{j-1}, \xi_{\mu}^{j-1}; \eta_{r}, \xi_{r}; \eta_{0}, \xi_{0})} + \sum_{a=1}^{p} \left[C_{a}(X) \log \frac{E_{a}(X; \eta_{\mu}^j, \xi_{\mu}^j)}{E_{a}(X; \eta_{\mu}^{j-1}, \xi_{\mu}^{j-1})} - C_{a}(X) \log \frac{E_{a}(X; \eta_{\mu}^j, \xi_{\mu}^j)}{E_{a}(X; \eta_{\mu}^{j-1}, \xi_{\mu}^{j-1})} \right] \right) \\
= \sum_{j=1}^{k} \sum_{\mu=1}^{\nu_j} a_{\mu} \int_{C_{j}} \int_{C_{j}} H_{\mu}(s, \eta, \xi) d\xi d\eta,
\]

where \( s \) is a root of \( q_2(s, \eta, \xi) = 0 \). Furthermore, the integral has always a finite value for each compact \( \mathcal{S}' \).

References


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