ON THE BLOCH-NEVANLINNA CONJECTURE

BY

P. L. DUREN (ANN ARBOR, MICHIGAN)

A function \( f(z) \) analytic in \(|z| < 1\) is said to be of **bounded characteristic** if

\[
\int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \leq M < \infty, \quad r < 1.
\]

It is well known that such a function has a (finite) radial limit almost everywhere.

The "Bloch-Nevanlinna conjecture" asserts that if \( f(z) \) is of bounded characteristic, so is \( f'(z) \). Many counterexamples have been given. The purpose of this note is to offer two simple constructions which disprove the conjecture in a decisive way. The arguments are based on the following known lemma concerning the Rademacher functions \( q_n(t) \):

**Lemma.** If \( \sum_{n=1}^{\infty} c_n z^n \) is analytic in \(|z| < 1\) and \( \sum_{n=1}^{\infty} |c_n|^2 = \infty \), then for almost every \( t \in [0, 1] \), the function \( \sum_{n=1}^{\infty} q_n(t) c_n z^n \) has a radial limit almost nowhere (i.e., on no set of positive measure).

Since the existence of a radial limit is equivalent to the Abel summability of the trigonometric series \( \sum q_n(t) c_n e^{inz} \), this lemma is a special case of a much more general result ([2], p. 214) asserting the non-summability of such series by any regular method of summability.

**Theorem 1.** There exists a function \( f(z) \) analytic in \(|z| < 1\) and continuous in \(|z| \leq 1\), such that no fractional derivative \( f^{(\alpha)}(z) \) of positive order has a radial limit on any set of positive measure.

If \( g(z) = \sum_{n=1}^{\infty} b_n z^n \), its fractional derivative of order \( \alpha \) will be defined as

\[
g^{(\alpha)}(z) = \sum_{n=1}^{\infty} \lambda_n b_n z^n,
\]
where
\[ \lambda_n = \frac{n!}{\Gamma(n+1-a)} \sim n^a. \]

This differs from the usual definition by an inessential factor \( s^a \).

To prove the theorem, let
\[ f_t(z) = \sum_{n=1}^{\infty} q_n(t) a_n z^n, \]
where \( q_n(t) \) is the \( n^{th} \) Rademacher function and
\[ a_n = \begin{cases} 1/k^2, & n = 2^k \ (k = 1, 2, \ldots), \\ 0 & \text{otherwise.} \end{cases} \]

For every \( t \in [0, 1] \), \( f_t(z) \) is continuous in the closed disk, since \( \Sigma |a_n| < \infty \). Let \( a_j = 1/j \), \( j = 1, 2, \ldots \). Then since \( \Sigma n^{2a} |a_n|^2 = \infty \) for every \( a > 0 \), it follows from the lemma that for each \( t \in [0, 1] \) outside a set \( E_j \) of measure zero, the fractional derivative \( f_t^{(a)}(z) \) has a radial limit almost nowhere.

We claim now that for each fixed \( t \not\equiv E = \bigcup_{j=1}^{\infty} E_j \), no fractional derivative \( f_t^{(a)}(z) \) of positive order \( a \) can have a radial limit on any set of positive measure. For if \( a_j < a \), \( f_t^{(a)}(z) \) must have a radial limit wherever \( f_t^{(a)}(z) \) does. To see this, we write
\[ h(r) = f_t^{(a)}(re^{i\theta}) = \sum_{n=1}^{\infty} c_n r^n, \]
where \( c_n = q_n(t) \lambda_n a_n e^{in\theta} \). Under the assumption that \( \lim_{r \to 1} h(r) \) exists we are to show that
\[ H(r) = \sum_{n=1}^{\infty} \frac{\Gamma(n+1-a)}{\Gamma(n+1-a_j)} c_n r^n \]
also tends to a finite limit as \( r \to 1 \). But \( H(r) \) may be written in the form
\[ H(r) = \frac{1}{\Gamma(\beta_j)} \int_0^1 t^{-a}(1-t)^{\beta_j-1} h(rt) \, dt, \]
where \( \beta_j = a - a_j > 0 \) and we assume without loss of generality that \( a < 1 \). Hence \( H(r) \) has a limit if \( h(r) \) does, by the Lebesgue bounded convergence theorem. This completes the proof.

Since the exceptional set \( E \) has measure zero, almost every choice of \( t \) gives a function \( f = f_t \) with the required property.
A similar construction shows that the Bloch-Nevanlinna conjecture may fail even if \( f(z) \) is very smooth on the boundary. We refer to the Zygmund class \( \Lambda^* \) which consists of all continuous functions \( F(\theta) \) periodic with period \( 2\pi \), such that

\[
|F(\theta + h) - 2F(\theta) + F(\theta - h)| \leq Ah
\]

for some constant \( A \) and all \( h > 0 \). Thus \( \Lambda^* \) contains the Lipschitz class \( \Lambda_1 \), and it can be shown that \( \Lambda^* \subset \Lambda_a \) for every \( a < 1 \). (See [2], p. 42 ff.)

**Theorem 2.** There exists a function \( f(z) \) analytic in \( |z| < 1 \) and continuous in \( |z| \leq 1 \), such that \( f(e^{i\theta}) \in \Lambda^* \), yet \( f'(z) \) has a radial limit almost nowhere.

We remark that the slightly stronger hypothesis \( f(e^{i\theta}) \in \Lambda_1 \) would imply \( f'(z) \) is bounded, thus that it has a radial limit almost everywhere. To prove Theorem 2, let

\[
a_n = \begin{cases} 
1/n, & n = 2^k \ (k = 1, 2, \ldots), \\
0 & \text{otherwise,}
\end{cases}
\]

and define \( f_t(z) = \sum_{n=1}^{\infty} p_n(t) a_n z^n \). Then

\[
\sum_{n=1}^{N} n^2 |a_n| = O(N),
\]

so \( f_t(e^{i\theta}) \in \Lambda^* \) for every \( t \). (See e.g. [1], p. 252.) But \( \sum n^2 |a_n|^2 = \infty \), so the lemma asserts that for almost every \( t \), \( f_t'(z) \) has a radial limit almost nowhere.

**References**


University of Michigan

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