PRIME NUMBERS SUCH THAT THE SUMS OF THE DIVISORS
OF THEIR POWERS ARE PERFECT SQUARES

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1. Introduction. For an integer \( n \geq 1 \) let \( \sigma(n) \) be the sum of the divisors of \( n \). For a fixed integer \( \alpha \geq 1 \) we consider the primes \( p \) for which

(1) \[ \sigma(p^\alpha) \text{ is a perfect square.} \]

Using \[ \sigma(p^\alpha) = 1 + p + \ldots + p^\alpha \] one shows easily that for \( \alpha = 1 \) this can happen only for \( p = 3 \) and that for no prime \( p \) the sum \( \sigma(p^2) \) is a square. Schinzel [3] showed that for \( \alpha = 3 \) we must have \( p = 7 \) and Thébault [5] considered \( \alpha = 4 \) showing that in this case \( p = 3 \).

On the other hand, by the celebrated general theorems of Siegel (see [2], Chapter 28, and [4]) and Baker [1] it is known that only finite number of primes satisfy (1) and, moreover, any such \( p \) satisfies

\[ p < \exp \exp \exp (\alpha^{10^3}). \]

We prove

Theorem. Let \( \alpha \) be an odd integer greater than or equal to 3. Then all prime numbers \( p \) such that \( \sigma(p^\alpha) \) is a perfect square satisfy \( p < 2^{2\alpha+1} \).

Numerical example. There is no prime \( p \) such that \( \sigma(p^5) \) is a perfect square.

2. Lemmas. Let \( \alpha \) be an integer greater than or equal to 3 and assume that \( \sigma(p^\alpha) = m^2 \). Then we have

\[ p(1 + p + \ldots + p^{\alpha-1}) = (m-1)(m+1). \]

Hence \( p \mid m+1 \) or \( p \mid m-1 \).

Lemma 1. There exists an integer \( x \neq 0 \) such that

\[ 2^6(1 + p + \ldots + p^{\alpha-3}) = p(px)^2 + p \{-2(2^2 - 1)px + (2^3 + 1)\} + \varepsilon \{-2^3 px + 2^3 (2^2 - 1)\} - 2^4 x, \]

where \( \varepsilon \) is +1 when \( p \mid m+1 \) and \( \varepsilon \) is −1 when \( p \mid m-1 \).
Proof. If \( p \mid m+1 \), then we write \( m+1 = px_1 \) (\( x_1 \geq 1 \)). We have
\[
p(1 + p + \ldots + p^{a-2}) = px_1^2 - (2x_1 + 1).
\]
If we write \( 2x_1 + 1 = px_2 \) (\( x_2 \geq 1 \)), then we have
\[
2^2 p(1 + p + \ldots + p^{a-3}) = p^2 x_2^2 - 2px_2 - (2^2x_2 + 2^2 - 1).
\]
If we write \( 2^2x_2 + 2^2 - 1 = px_3 \) (\( x = x_3 \geq 1 \)), then we obtain the formula of the lemma. If \( p \mid m-1 \), then we put \( m-1 = px_1 \), \( 2x_1 - 1 = px_2 \), and \( 2^2x_2 + 1 - 2^2 = -px \).

Remark. By Lemma 1, all primes \( p \) satisfying (1) are odd.

Lemma 2. If \( 3 \leq k \leq \alpha \), then the equation
\[
2^{\beta(k)}(1 + p + \ldots + p^{a-k}) = p^{k-2}(px_k)^2 + \sum_{i=2}^{k} p^{k-i} \{\gamma_i^{(k)} px_k + \delta_i^{(k)}\} - 2^{\mu(k)} x_k
\]
holds, where \( \beta(k), \gamma_i^{(k)}, \delta_i^{(k)}, \mu(k) \) are integers that depend only on \( k \) and are independent of \( \alpha, p, m \). Moreover, \( \beta(k), \mu(k) > 0 \), and \( x_k \) is an integer.

Proof. For \( \alpha = 3 \) this is Lemma 1. Assume that the lemma holds for \( 3 \leq k < \alpha \). Then, from
\[
2^{\beta(k)} p(1 + p + \ldots + p^{a-(k+1)})
\]
\[
= p^{k-2}(px_k)^2 + \sum_{i=2}^{k-1} p^{k-i} \{\gamma_i^{(k)} px_k + \delta_i^{(k)}\} + \gamma_k^{(k)} px_k - (2^{\mu(k)} x_k + 2^{\beta(k)} - \delta_k^{(k)})
\]
we see that \( p \mid 2^{\mu(k)} x_k + 2^{\beta(k)} - \delta_k^{(k)} \). If we write
\[
2^{\mu(k)} x_k + 2^{\beta(k)} - \delta_k^{(k)} = px \quad (x_{k+1} = x),
\]
then
\[
2^{\beta(k) + 2^{\mu(k)}}(1 + p + \ldots + p^{a-(k+1)})
\]
\[
= p^{k-1}(px)^2 + p^{k-1} \{ -2(2^{\beta(k)} - \delta_k^{(k)}) px + (2^{\beta(k)} - \delta_k^{(k)})^2 \} +
\]
\[
+ p^{k-2} \{ 2^{\mu(k)} \gamma_2^{(k)} px - 2^{\mu(k)} \gamma_2^{(k)} (2^{\beta(k)} - \delta_k^{(k)}) \} +
\]
\[
+ p^{k-3} \{ 2^{\mu(k)} \gamma_3^{(k)} px + \{ -2^{\mu(k)} \gamma_3^{(k)} (2^{\beta(k)} - \delta_k^{(k)}) + 2^{2^{\mu(k)}} \delta_2^{(k)} \} \} + \ldots +
\]
\[
+ 2^{\mu(k)} \gamma_k^{(k)} px + \{ -2^{\mu(k)} \gamma_k^{(k)} (2^{\beta(k)} - \delta_k^{(k)}) + 2^{2^{\mu(k)}} \delta_k^{(k)} \} - 2^{2^{\mu(k)}} x.
\]

Now we consider the properties of the constants \( \beta(k), \gamma_i^{(k)}, \delta_i^{(k)}, \mu(k) \) constructed in the proof of Lemma 2. By the proof of Lemma 2 and Lemma 1, we have

Lemma 3. If \( 3 \leq k \leq \alpha \), then
\[
\mu(k) = 2^{k-1}, \quad \beta(k) = 2^k - 2,
\]
\[ \gamma_2^{(k+1)} = -2(2^{\beta(k)} - \delta_2^{(k)}), \]
\[ \gamma_i^{(k+1)} = 2^{\mu(k)} \gamma_i^{(k)}, \quad i = 3, 4, \ldots, k + 1, \]
\[ \delta_2^{(k+1)} = (2^{\beta(k)} - \delta_2^{(k)})^2, \quad \delta_3^{(k+1)} = -2^{\mu(k)} \gamma_2^{(k)} (2^{\beta(k)} - \delta_2^{(k)}), \]
\[ \delta_i^{(k+1)} = -2^{\mu(k)} \gamma_i^{(k)} (2^{\beta(k)} - \delta_2^{(k)}) + 2^{\mu(k)} \delta_i^{(k+1)}, \quad i = 3, 4, \ldots, k. \]

**Lemma 4.** If \( 3 \leq k \leq \alpha \) and \( 2 \leq i \leq k \), then
\[ |\gamma_i^{(k)}| < 2^{1+2^k} \quad \text{and} \quad |\delta_i^{(k)}| < 2^{2^{k+1} - 1}. \]

**Proof.** By the proof of Lemma 1 we have
\[ \beta(3) = 6, \quad \mu(3) = 2^2, \quad \gamma_2^{(3)} = -2(2^2 - 1), \quad \gamma_3^{(3)} = \pm 2^3, \]
\[ \delta_2^{(3)} = 1 + 2^3, \quad \delta_3^{(3)} = \pm 2^3 (2^2 - 1). \]
Thus, for \( k = 3 \) the lemma holds. Let \( k > 3 \) and assume that the lemma holds for \( k \). Then, by Lemma 3, we have
\[ |\gamma_2^{(k+1)}| < 2(2^{2^k} + 2^{1+2^k} + \ldots + 2^k) < 2^{1+2^k}, \]
\[ |\gamma_i^{(k+1)}| < 2^{2^k - 1} \cdot 2^{1+2^k} < 2^{1+2^k+1} \quad (3 \leq i \leq k + 1), \]
\[ |\delta_2^{(k+1)}| < (2^{2^k} + 2^{1+2^k} + \ldots + 2^k)^2 < 2^{1+2^k+1}, \]
\[ |\delta_i^{(k+1)}| < 2^{2^k - 1} \cdot 2^{1+2^k} (2^{2^k} + 2^{1+2^k} + \ldots + 2^k) + 2^{2^k} \cdot 2^{1+2^k+1} \]
\[ < 2^{1+2^k+1} \quad (3 \leq i \leq k + 1). \]

**Lemma 5.** If \( \alpha \) is an odd integer greater than or equal to 3 and \( p > 2^{2\alpha+1} \),
then \( x_k \neq 0 \) for \( k = 3, 4, \ldots, \alpha \).

**Proof.** Let \( k \) be the first \( r \) such that \( x_r = 0 \). By Lemma 1 we have \( k > 3 \). From Lemma 2 we obtain
\[ 2^{\beta(k)} (1 + p + \ldots + p^{\alpha-k}) = \sum_{i=2}^{k} \delta_i^{(k)} p^{k-i}. \]
Since \( \alpha \) is odd, we have \( \alpha - k \neq k - 2 \).

**Case 1.** \( \alpha - k > k - 2 \). By Lemma 4, we have \( |\delta_1^{(k)}| < p/2 \). Hence \( 2^{\beta(k)} = \delta_2^{(k)} = \ldots = \delta_2^{(k)} \) and the coefficient \( 2^{\beta(k)} \) of \( p^{\alpha-k} \) is zero, a contradiction.

**Case 2.** \( \alpha - k < k - 2 \). Since \( \delta_2^{(k)} = 0 \), by Lemma 3 we have \( 2^{\beta(k+1)} = \delta_i^{(k-1)} \) for \( i = 2, 3, \ldots, k \). Hence, by Lemma 2, we get
\[ (2) \quad 2^{\beta(k+1)} (p + p^2 + \ldots + p^{2^{(k-1)}}) \]
\[ = p^{k-3} (px_{k-1})^2 + \sum_{i=2}^{k-2} p^{k-1-i} (\gamma_i^{(k-1)} px_{k-1} + \delta_i^{(k-1)}) + \]
\[ + \gamma_i^{(k-1)} px_{k-1} - 2^{\mu(k-1)} x_{k-1}. \]
Consequently, \( p \mid x_{k-1} \). If \( x_{k-1} \neq 0 \), then expressing both sides of (2) in the\footnote{\textsuperscript{a}} form \( \sum_{i} a_i p^i \) (\( |a_i| < p/2 \)) we see that the highest power of \( p \) on the right-hand side is \( \alpha - (k - 1) < k - 1 \) while the one on the left-hand side is equal to or greater than \( k - 3 + 2 + 2 = k + 1 \), a contradiction. Hence \( x_{k-1} = 0 \), which contradicts the choice of \( k \).

3. Proof of the Theorem. Let \( \alpha \geq 3 \) and \( p > 2^{2^\alpha + 1} \). Assume \( \sigma(p^\alpha) = m^2 \). If we put \( k = \alpha \) in the formula of Lemma 2, we have

\[
p^\alpha x_\alpha^2 + \left( \sum_{i=2}^{\alpha} p^{2i-1} \gamma(\alpha) - 2^{\mu(\alpha)} \right) x_\alpha + \left( \sum_{i=2}^{\alpha} p^{2i-1} \delta_i(\alpha) - 2^{\epsilon(\alpha)} \right) = 0.
\]

By Lemma 4, the discriminant \( D \) of the quadratic equation (3) satisfies

\[
|D| \leq \left( \sum_{i=2}^{\alpha} p^{2i-1} \cdot 2^{2^\alpha + 2} \right)^2 + 2^2 p^\alpha \left( \sum_{i=2}^{\alpha} p^{2i-1} \cdot 2^{2^\alpha + 1} \right) \leq \alpha^2 p^{2(\alpha - 1)} \cdot 2^{2^\alpha + 4} + 5.
\]

Hence the root \( x_\alpha \) of (3) satisfies \( |x_\alpha| < \alpha \cdot 2^{2^\alpha + 4} < 1 \). By Lemmas 2 and 5, this is a contradiction.

4. Numerical example. If \( \sigma(p^5) = m^2 \), then \( p \mid m+1 \) or \( p \mid m-1 \).

Case 1. \( p \mid m+1 \). If we put \( m+1 = px_1 \), \( 2x_1 + 1 = px_2 \), \( 4x_2 + (2+1) = px_3 \), \( 2x_3 + 2^2 + 1 = px_4 \), and \( 2^3 x_4 + (2^5 + 2^2 - 1) = px \), then \( x \) is an integer not equal to 0 and

\[
f_1(x) = p^3(px)^2 + p^3 \left[ -2(1+2+2^5)px + (1+2^3+2^6+2^7+2^{10}) \right] +
+ p^2 \left[ -2^4(2^2+1)px + 2^4(1+2+2^2+2^3+2^5+2^7) \right] +
+ p \left[ -2^5(2^2-1)px + 2^5(1+2+2^3+2^4+2^5) \right] -
- 2^7px - 2^8x - 2^7(2^6-1) = 0.
\]

The value \( x_0 \) such that \( y = f_1(x_0) \) has the minimal value is given by

\[
x_0 = (1+2+2^5)/p + 2^3(2^2+1)/p^2 + 2^4(2^2-1)/p^3 + 2^6/p^4 + 2^7/p^5 > 0.
\]

If \( p > 2^6 = 64 \), then \( x_0 < 2^6/p < 1 \). Since \( f_1(2^6/p) > 0 \), we have \(-1 < x < 1 \). Therefore, in this case, if \( p > 2^6 \), then \( \sigma(p^\alpha) \) cannot be a perfect square.

Case 2. \( p \mid m-1 \). If we put \( m-1 = px_1 \), \( 2x_1 - 1 = px_2 \), \( 2^2 x_2 + 1 - 2^2 = -px_3 \), \( 2x_3 + 2^3 + 2^2 - 1 = px_4 \), and \( 2^3 x_4 + (2^6 + 2^5 + 2^2 - 1) = px \), then \( x \) is an integer not equal to 0 and

\[
f_2(x) = p^3(px)^2 + p^3 \left[ -2(2^6+2^5+2^2-1)px + (2^6+2^5+2^2-1)^2 \right] +
+ p^2 \left[ -2^4(2^3+2^2-1)px + 2^4(2^{10}+2^6+1) \right] +
+ p \left[ -2^5(2^2-1)px + 2^5(2^9+2^5-2^3+2+1) \right] +
+ 2^7px - 2^7(2^5+1)-2^8x = 0.
\]
The value \( x_0 \) such that \( y = f_2(x_0) \) has the minimal value is given by
\[
x_0 = \frac{(2^6 + 2^5 + 2^2 - 1)/p + 2^3 (2^3 + 2^2 - 1)/p^2 + 2^4 (2^2 - 1)/p^3 - 2^6/p^4 + 2^7/p^5}{p^6} > 0.
\]
If \( p > 2^6 + 2^5 + 2^2 \), then \( x_0 < (2^6 + 2^5 + 2^2)/p < 1 \). Since \( f_2 ((2^6 + 2^5 + 2^2)/p) \) > 0, we have \(-1 < x < 1\). Therefore, in this case, the primes \( p \) such that \( \sigma(p^5) \) is a perfect square satisfy \( p < 2^6 + 2^5 + 2^2 = 100 \).

From cases 1 and 2 we infer that the primes \( p \) such that \( \sigma(p^5) \) is a perfect square satisfy \( p < 100 \). Using a computer, one checks that the values \( 1 + p + p^2 + p^3 + p^4 + p^5 \) are not perfect squares for \( 3 \leq p \leq 97 \).

REFERENCES


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*Reçu par la Rédaction le 30. 7. 1980*