Four different unknown functions satisfying the triangle mean value property for harmonic polynomials, II

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Abstract. If $f_j$ : $C \rightarrow R$ satisfy the quasi-triangle mean value property $f_0(x + y) + f_1(x + \theta y) + + f_2(x + \theta^2 y) = 3f_3(x)$ for all $x, y \in C$, $\theta = \exp(2\pi i/3)$, then there exist generalized quadratic polynomials such that $f_j(x) = Q_j^0 + Q_j^1(x) + Q_j^2(x)$ for all $x \in C$. In addition if $f_j$ are bounded on a set of positive Lebesgue measure, then $f_j$ are given by harmonic polynomials of degree $\leq 2$.

3. Reduction to generalized quadratic polynomials. In the previous note (*) we found the continuous solution of equation (M). In this note it is our purpose to prove the following two extensions of a theorem proved in (*).

One of them is:

THEOREM 1. Let $R$ be the set of all real numbers and let $C$ be the set of all complex numbers. If $f_j$ $(j = 0, 1, 2, 3)$ : $C \rightarrow R$ satisfy the quasi-triangle mean value property

$$(M) \quad \sum_{j=0}^{2} f_j(x + \theta^j y) = 3f_3(x)$$

for all $x, y \in C$, where $\theta$ is a number $\exp(2\pi i/3)$, then there exist generalized quadratic polynomials (*) such that

$$(3.1) \quad f_j(x) = Q_j^0 + Q_j^1(x) + Q_j^2(x)$$

for all $x \in C$ and for each $j = 0, 1, 2, 3$, where

(i) $Q_j^0$ are real constants,

(ii) $Q_j^1$ : $C \rightarrow R$ are additive functions and

(iii) $Q_j^2$ : $C \rightarrow R$ are symmetric bi-additive functions, i.e., $Q_j^2(x) = Q_j^2(x, x)$ and $Q_{j,2}(x, y) : C \times C \rightarrow R$ are symmetric bi-additive.

The other is:

THEOREM 2. Let $\mu$ denote Lebesgue measure on $R \times R$ and let $\Omega \subset R \times R$ be a measurable subset with $\mu(\Omega) > 0$. Let the functions $f_j(x) \equiv u_j(x_1, x_2)$:


(*) See [1], [4].
$\mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$, $j = 0, 1, 2, 3$, be solutions of (M). Then $f_j$ are given by (3.1) for all $x \in \mathcal{C}$. In addition, if $u_j$ are bounded on $\Omega$, then $u_j$ are continuous everywhere and are harmonic polynomials of degree $\leq 2$.

4. Proof of Theorem 1. Rewrite (M) as

$$(4.1) \quad (T_0^2 + T_1^{\theta_2} + T_2^{\theta_2^2} - 3T_3^0)f(x) = 0$$

for all $x, y \in \mathcal{C}$, where the shift operator is previously defined by

$$T_j^0f(x) = f_j(x) \quad \text{and} \quad T_j^yf(x) = f_j(x + y)$$

for each $j = 0, 1, 2, 3$ and for all $x, y \in \mathcal{C}$.

Replace $x$ by $x - \theta z_1$ and $y$ by $y + z_1$ in (4.1). Then clearly

$$(4.2) \quad (T_0^{(1-\theta)z_1+y} + T_1^{\theta y} + T_2^{\theta^2y + (\theta^2 - \theta)z_1} - 3T_3^{-\theta z_1})f = 0$$

for all $x, y, z_1 \in \mathcal{C}$.

If the difference operator $\Delta$ is defined by

$$\Delta_yf_j(x) = (T_j^y - T_j^0)f(x) \quad \text{for all} \quad x, y \in \mathcal{C},$$

then by taking a difference of (4.2) and (4.1) one obtains

$$(4.3) \quad \Delta_{(1-\theta)z_1} f_0(x + y) + \Delta_{\theta^2 - \theta z_1} f_2(x + \theta^2 y) - 3\Delta_{-\theta z_1} f_3(x) = 0.$$ 

Similarly, by replacing $x$ by $x - \theta^2 z_2$ and $y$ by $y + z_2$ in (4.3), we infer

$$(4.4) \quad \Delta_{(1-\theta)z_1} f_0(x + y + (1 - \theta^2)z_2) + \Delta_{\theta^2 - \theta z_1} f_2(x + \theta^2 y) - 3\Delta_{-\theta z_2} f_3(x - \theta^2 z_2) = 0$$

for all $x, y, z_1, z_2 \in \mathcal{C}$, which with (4.3) implies

$$(4.5) \quad \Delta_{(1 - \theta)z_1} \Delta_{(1 - \theta^2)z_2} f_0(x + y) - 3\Delta_{-\theta z_1} \Delta_{-\theta^2 z_2} f_3(x) = 0.$$ 

Finally, set $y = z_3$ in (4.5). Then

$$\Delta_{(1 - \theta)z_1} \Delta_{(1 - \theta^2)z_2} f_0(x + z_3) - 3\Delta_{-\theta z_1} \Delta_{-\theta^2 z_2} f_3(x) = 0$$

which with (4.5) implies the desired equation

$$\Delta_{(1 - \theta)z_1} \Delta_{(1 - \theta^2)z_2} \Delta_{z_3} f_0(x) = 0$$

for all $x, z_1, z_2, z_3 \in \mathcal{C}$, and therefore

$$(4.6) \quad \Delta_{z_3}^2 f_0(x) = 0 \quad \text{for all} \quad x, z_3 \in \mathcal{C}.$$ 

Hence it immediately follows from the results of S. Mazur and W. Orlicz [3] with equation (4.6) that there exists a quadratic polynomial such that

$$(4.7) \quad f_0(x) = Q_0^5 + Q_1^5(x) + Q_2^5(x) \quad \text{for all} \quad x \in \mathcal{C},$$

where functions $Q_j^5$, $j = 0, 1, 2, 3$, are defined in Theorem 1.
In view of equation (M) it is clear that obvious modifications can be repeated for the terms \( f_1, f_2 \) and \( f_3 \) to obtain
\[
\Delta^j \phi_j(x) = 0 \quad \text{for each } j = 1, 2, 3 \text{ and for all } x, u \in \mathbb{C},
\]
since \( \theta^m \neq \theta^p, \ m \neq p \) for \( m, p = 1, 2, 3 \). Theorem 1 is proved.

5. Proof of Theorem 2. By (4.7) and the additivity of \( Q_i^0 \) we obtain ([3])

\[
f_0(Nx) = Q_0^0 + NQ_0^1(x) + N^2 Q_0^2(x) \quad \text{for } N = 1, 2, 3.
\]

System (5.1) is clearly solved for \( Q_0^j, j = 0, 1, 2, \) in terms of \( f_0(Nx), \)
\( N = 1, 2, 3, \) since the Vandermonde determinant
\[
\begin{vmatrix}
1 & 1 & 1 \\
1 & 2 & 2^2 \\
1 & 3 & 3^2
\end{vmatrix} \neq 0.
\]

But \( f_0(x) \equiv |u_0(x_1, x_2)| \) is bounded for all \( (x_1, x_2) \in \Omega \). Hence \( Q_0^0, Q_0^1(x) \) and \( Q_0^2(x) \) for all \( x \) are bounded on a set of positive Lebesgue measure. Further, the identity
\[
Q_{0,2}(x, y) = \frac{1}{2} (Q_{0,2}(x + y, x + y) - Q_{0,2}(x - y, x - y))
\]
for all \( x, y \in \mathbb{C} \) shows that \( Q_{0,2}(x, y) \) is also bounded on a set of positive Lebesgue measure, since \( Q_{0,2}(x + y, x + y) = Q_0^0(x + y) \) and \( Q_0^2(x - y) \) are bounded. If one briefly defines \( Q_{0,2}(x, y) = Q_{0,2}(x_1, x_2, x_3, x_4) \), then by the bi-additivity of \( Q_{0,2} \) one readily obtains

\[
Q_{0,2}(x_1, x_2, x_3, x_4) = a_1(x_1, x_3) + a_2(x_1, x_4) + a_3(x_2, x_3) + a_4(x_2, x_4),
\]

where \( a_1, a_2, a_3, a_4: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) are additive in the first and second variables separately. Moreover, (5.2) implies
\[
a_1(x_1, x_3) = Q_{0,2}(x_1, x_2, x_3, x_4) + Q_{0,2}(x_1, -x_2, x_3, -x_4) +
+ Q_{0,2}(x_1, -x_2, x_3, x_4) + Q_{0,2}(x_1, x_2, x_3, -x_4).
\]

This shows that \( a_1 \) is bounded. Similarly, \( a_2, a_3, a_4 \) are bounded. It now follows by well-known theorems of additive functions (2) that \( Q_0^j, j = 1, 2, 3, \) must be continuous everywhere and, by (5.1), so is \( f_0. \) Similarly, if \( |u_j| < k, \ k > 0, \ j = 0, 1, 2, \) for all \( (x_1, x_2) \in \Omega, \) then \( f_1, f_2, f_3 \) are also continuous everywhere. Hence the theorem in (*) immediately implies that \( f_j, j = 0, 1, 2, 3, \) are given by harmonic polynomials of degree \( \leq 2. \) This proves Theorem 2.

6. Corollaries. As consequences of Theorem 1 and Theorem 2, we obtain the following corollaries.

(2) See [2].
Corollary 1. A function \( f : C \to R \) or \( C \) satisfies the triangle mean value property

\[
\sum_{j=0}^{2} f(x + \theta^j y) = 3f(x) \quad \text{for all } x, y \in C
\]

if and only if there exists a generalized quadratic polynomial such that

\[
f(x) = Q^0 + Q^1(x) + Q^2(x) \quad \text{for all } x \in C,
\]

where \( Q^0 \) is a real or a complex constant and an additive function \( Q^1 : C \to R \) or \( C \) and a symmetric bi-additive function \( Q^2 : C \to R \) or \( C \) must satisfy the equation

\[
\sum_{j=0}^{2} [Q^1(\theta^j y) + 2Q_2(x, \theta^j y) + Q^2(\theta^j y)] = 0 \quad \text{for all } x, y \in C.
\]

Proof. By Theorem 1 we obtain (6.2). Substitute (6.2) in (6.1) to obtain (6.3).

Corollary 2. Let \( f : C \to C \). Then the measurably bounded solution of (6.1) is given by a complex polynomial such that

\[
f(x) = a_0 + a_1 x + a_2 \bar{x} + a_3 x^2 + a_4 x \bar{x} + a_5 \bar{x}^2
\]

where \( a_k, k = 0, 1, \ldots, 5 \), are complex constants.

Proof. If \( f : C \to C \) satisfies (6.1), then \( f \) is given by (6.2). It readily follows from a similar proof of Theorem 2 that \( Q^1, Q^2 : C \to C \) are also bounded on a set of positive Lebesgue measure and hence are continuous everywhere. The continuous additive function \( Q^1 : C \to C \) is given by \( Q^1(x) = ax + b\bar{x} \) for all \( x \in C \) where \( \bar{x} \) denotes the conjugate of \( x \). Hence, by the bi-additivity of \( Q^2 \), we obtain

\[
Q_2(x_1, x_2) = a(x_2)x_1 + b(x_2)\bar{x}_1
\]

for all \( x_1, x_2 \in C \), and \( a, b : C \to C \). But by the symmetry of \( Q_2 \) we have

\[
a(x_2)x_1 + b(x_2)\bar{x}_1 = a(x_1)x_2 + b(x_1)\bar{x}_2.
\]

Set \( x_1 = 1 \) and \( x_1 = i \) in (6.6) to obtain

\[
a(x_2) + b(x_2) = a(1)x_2 + b(1)\bar{x}_2
\]

and \( a(x_2)i - b(x_2)i = a(i)x_2 + b(i)\bar{x}_2 \) which implies

\[
-a(x_2) + b(x_2) = a(i)x_2 + b(i)i\bar{x}_2.
\]

If we solve equations (6.7) and (6.8) for \( a \) and \( b \), then \( b(x_2) = c_1 x_2 + c_2 \bar{x}_2 \) and \( a(x_2) = c_3 x_2 + c_4 \bar{x}_2 \) with complex constants \( c_1, c_2, c_3 \) and \( c_4 \), which with (6.5) yield \( Q^2(x) = a_3 x^2 + a_4 x\bar{x} + a_5 \bar{x}^2 \). Hence we obtain (6.4).

Conversely, (6.4) satisfies (6.1).

Work supported by NRC grant A. 2972.
References


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Reçu par la Rédaction le 16. 03. 1978