A CHARACTERIZATION
OF COMPLETE BI-BROUWERIAN LATTICES

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1. In the paper [1] Abian proved that a Boolean ring is complete if and only if any 2-solvable system of one-variable Boolean polynomial equations is solvable.

The purpose of this note is to characterize those lattices in which every 2-solvable system of one-variable lattice polynomial equations is solvable.

The main result is the following

THEOREM. A lattice has the property that any 2-solvable system of one-variable lattice equations is solvable if and only if it is complete and bi-Brouwerian.

2. One-variable lattice polynomials are defined inductively as follows:
   (i) \( p_0 = a, p_1 = x \) are lattice polynomials.
   (ii) If \( p, q \) are lattice polynomials in one variable \( x \), say, then so are \( p \lor q, p \land q \).

A polynomial equation in \( x \) over a lattice \( \mathcal{L} \) is an expression of the form \( p = q \), where \( p \) and \( q \) are one-variable lattice polynomials in \( x \) over \( \mathcal{L} \).

A set \( \Sigma \) of polynomial equations in \( x \) is said to be 2-solvable if every subsystem of \( \Sigma \) consisting of two equations is solvable.

A Brouwerian lattice is a lattice \( \mathcal{L} \) in which, for every \( a, b \in \mathcal{L} \), there exists \( a \ast b \in \mathcal{L} \) such that

\[
    a \land x \leq b \iff x \leq a \ast b.
\]

A bi-Brouwerian lattice is a Brouwerian lattice whose dual is Brouwerian, i.e., for every \( a, b \in \mathcal{L} \), there exists \( a \ominus b \in \mathcal{L} \) such that

\[
    a \lor x \geq b \iff x \geq a \ominus b.
\]

Any Brouwerian lattice is distributive.
3. Let $S$ be any non-empty subset of a lattice $\mathcal{L}$ having the property that any system of 2-solvable lattice polynomial equations in one variable is solvable.

Consider the system $\Sigma_0$ of equations $s \wedge x = s \ (s \in S)$. Clearly, every 2-equation subsystem $s_1 \wedge x = s_1$ and $s_2 \wedge x = s_2$ has the solution $x = s_1 \vee s_2$ and so $\Sigma_0$ is solvable; any solution being an upper bound for $S$. Let $U_0$ be the set of all solutions of $\Sigma_0$ and consider the system $\Sigma_1$ of equations

$$s \wedge x = s \quad (s \in S),$$
$$u \vee x = u \quad (u \in U_0).$$

Clearly, a 2-equation subsystem of the form

$$s_1 \wedge x = s_1,$$
$$u_1 \vee x = u_1$$

has the solution $x = u_1$, while a 2-equation subsystem of the form

$$u_1 \vee x = u_1,$$
$$u_2 \vee x = u_2$$

has the solution $x = u_1 \wedge u_2$.

Consequently, $\Sigma_1$ is 2-solvable and, therefore, $\Sigma_1$ has a solution which is, obviously, the least upper bound of $S$. Hence, $\mathcal{L}$ is join-complete and, specializing to the case where $S = \mathcal{L}$, has a greatest element 1. Similarly, we can show that $\mathcal{L}$ is meet-complete and has a least element 0. Now, let $a, b \in \mathcal{L}$, $U_{ab} = \{x \in \mathcal{L}; a \wedge x \leq b\}$ and consider the system $\Sigma_2$ of equations

$$b \vee (a \wedge x) = b,$$
$$u \wedge x = u \quad (u \in U_{ab}).$$

Clearly, a 2-equation subsystem of the form $b \vee (a \wedge x) = b$ and $u \wedge x = u$ has the solution $x = u$, while a 2-equation subsystem of the form $u_1 \wedge x = u_1$ and $u_2 \wedge x = u_2$ has the solution $x = u_1 \vee u_2$.

Consequently, $\Sigma_2$ is 2-solvable and, therefore, $\Sigma_2$ has a solution which is the largest solution of the inequality $a \wedge x \leq b$. Thus, $\mathcal{L}$ is Brouwerian. Similarly, the dual of $\mathcal{L}$ is Brouwerian so that $\mathcal{L}$ is bi-Brouwerian.

4. In preparation for the proof of the converse we state

**Lemma 1.** In a bi-Brouwerian lattice, the following, together with their duals, hold:

(i) $x \leq y \iff x \div y = 1$,
(ii) $y \leq x \div y$,
(iii) $x \div (y \div z) = (x \div y) \div z$, 

(iv) \( x \ast (y \land z) = (x \ast y) \land (x \ast z) \),
(v) \( (x \lor y) \ast z = (x \ast z) \lor (y \ast z) \).

Proof. All of these results are proved in [3] except (iii) which is proved in [5].

**Lemma 2.** A complete lattice \( \mathcal{L} \) is Brouwerian if and only if
\[
\bigwedge_a (a \land a_a) = \bigvee_a (a \land a_a)
\]
for any non-empty subset \( \{a_a\}_{a \in I} \) of \( \mathcal{L} \).

**Lemma 3.** Any one-variable lattice polynomial \( p(x) \) in a distributive lattice with 0 and 1 can be uniquely expressed in the normal form
\[
p(x) = p_0 \lor (p_1 \land x) \quad \text{with} \quad p_0 \leq p_1.
\]
The proofs of Lemmas 2 and 3 are well known and may be found in [2] and [4], respectively.

If \( a \leq b \) in \( \mathcal{L} \), then the interval \([a, b]\) is the set \( \{x \in \mathcal{L}; a \leq x \leq b\} \).

In a Brouwerian lattice we write \( a \times b \) for \((a \ast b) \land (b \ast a)\), and in a lattice whose dual is Brouwerian we write \( a + b \) for \((a - b) \lor (b - a)\).

The following result is crucial:

**Lemma 4.** If \( \mathcal{L} = \langle L; \lor, \land, \ast, \neg; 0, 1 \rangle \) is a bi-Brouwerian lattice, then necessary and sufficient conditions for the existence of a solution of the pair \( \Sigma \) of lattice polynomial equations \( p(x) = q(x) \) and \( s(x) = t(x) \), where \( p, q, s, t \) are in the normal form, are the following:
\[
(*) \quad p_0 \leq p_1, \quad p_0 \leq q_1, \quad t_0 \leq s_1, \quad s_0 \leq t_1
\]
and
\[
p_0 + q_0 \leq s_1 \times t_1, \quad s_0 + t_0 \leq p_1 \times q_1.
\]
Moreover, if the pair \( \Sigma \) is solvable, then the solution set is the interval
\[
[(p_0 + q_0) \lor (s_0 + t_0), (p_1 \times q_1) \land (s_1 \times t_1)].
\]

Proof. For both equalities to hold it is necessary and sufficient that
\[
\{p(x) + q(x)\} \lor \{s(x) + t(x)\} = 0.
\]
Now, by the distributivity of \( \mathcal{L} \),
\[
p(x) - q(x) = 0 \iff \{p_1 \land (p_0 \lor x)\} - q(x) = 0
\]
\[
\iff \{p_1 - q(x)\} \lor \{(p_0 \lor x) - q(x)\} = 0
\]
\[
\iff q(x) \leq p_1 \text{ and } q(x) \leq p_0 \lor x.
\]
However,
\[
q(x) \leq p_1 \iff q_1 \land (q_0 \lor x) \leq p_1
\]
\[
\iff q_0 \lor x \leq q_1 \ast p_1
\]
\[
\iff q_0 \leq p_1 \text{ and } x \leq q_1 \ast p_1.
\]
Furthermore,
\[ q(x) \leq p_0 \lor x \iff p_0 - q(x) \leq x \]
\[ \iff p_0 - \{q_0 \lor (q_1 \land x)\} \leq x \]
\[ \iff (p_0 - q_0) \lor \{p_0 - (q_1 \land x)\} \leq x \]
\[ \iff p_0 - q_0 \leq x \text{ and } p_0 - (q_1 \land x) \leq x \]
which, since \( p_0 - (q_1 \land x) \leq q_1 \land x \leq x \), is equivalent to \( p_0 - q_0 \leq x \). Consequently,
\[ p(x) - q(x) = 0 \iff q_0 \leq p_1 \text{ and } p_0 - q_0 \leq x \leq q_1 \land p_1, \]
and, therefore,
\[ p(x) + q(x) = 0 \iff q_0 \leq p_1, \; p_0 \leq q_1 \text{ and } p_0 + q_0 \leq x \leq p_1 \times q_1. \]

Hence, \( p(x) = q(x) \) and \( s(x) = t(x) \) if and only if conditions (*) hold and
\[ (p_0 + q_0) \lor (s_0 + t_0) \leq x \leq (p_1 \times q_1) \land (s_1 \times t_1). \]

Finally, if conditions (*) hold, then
\[ p_0 - q_0 \leq q_0 \leq p_1 \leq q_1 \land p_1 \text{ and } p_0 - q_0 \leq q_0 \leq q_1 \leq p_1 \land q_1 \]
so that \( p_0 - q_0 \leq p_1 \times q_1. \)

Similarly, \( q_0 - p_0 \leq p_1 \times q_1 \) and, consequently, \( p_0 + q_0 \leq p_1 \times q_1. \) In a similar fashion we can show that \( s_0 + t_0 \leq s_1 \times t_1. \) Therefore,
\[ (p_0 + q_0) \lor (s_0 + t_0) \leq (p_1 \times q_1) \land (s_1 \times t_1) \iff p_0 + q_0 \leq s_1 \times t_1 \]
and
\[ s_0 + t_0 \leq p_1 \times q_1, \]
completing the proof of Lemma 4.

Now we are in a position to prove the sufficiency.

Let \( \mathcal{L} \) be a complete bi-Brouwerian lattice and let \( \Sigma: p_i(x) = q_i(x), \)
\( i \in I, \) be a system of polynomial equations, in one variable, which is 2-solvable. We may suppose, since \( \mathcal{L} \) is distributive, that each \( p_i \) is in its normal form
\[ p_i(x) = p_{i0} \lor (p_{i1} \land x) \quad \text{with } p_{i0} \leq p_{i1}. \]

Now, if \( j \) is an arbitrary member of the index set \( I, \) then, since \( \Sigma \)
is 2-solvable, it follows from Lemma 4 that \( p_j(x_i) = q_j(x_i) \) for each \( i \in I, \)
where
\[ x_i = (p_{j0} + q_{j0}) \lor (p_{j0} + q_{j0}). \]

We deduce that
\[ p_{j0} \lor \bigvee_{i \in I} (p_{j1} \land x_i) = q_{j0} \lor \bigvee_{i \in I} (q_{j1} \land x_i) \]
or, equivalently, from Lemma 2,

\[ \bigvee_{i \in I} (p_{j0} \lor (p_{j1} \land \bigvee_{i \in I} x_i)) = \bigvee_{i \in I} (q_{j0} \lor (q_{j1} \land \bigvee_{i \in I} x_i)). \]

Consequently, the equation \( p_j(x) = q_j(x) \) has the solution

\[ x = \bigvee_{i \in I} x_i = \bigvee_{i \in I} (p_{i0} + q_{i0}) \]

and, therefore, since \( j \) is arbitrary, it follows that \( \Sigma \) has a solution, namely

\[ x = \bigvee_{i \in I} (p_{i0} + q_{i0}). \]

REFERENCES


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