ON CLASSES OF ALGEBRAS
DEFINABLE BY REGULAR EQUATIONS

BY

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1. In [3] J. Płonka investigated the smallest class of algebras which contains a given equational class and is defined by regular equations. Especially he considered the connection between such a class and the formation of sums of direct systems of algebras defined in [2] under the assumption that all algebras have no nullary fundamental operations.

In this note we shall give a complete description of such a closure defined by regular equations for arbitrary classes of algebras and without any assumption on the fundamental operations (Theorem 1). This will lead to a characterization of the corresponding closure operator using the notion of the sum of a direct system in the case that no nullary fundamental operations occur (Theorem 2).

2. All algebras under consideration are of finitary type $\Delta = (n_{i})_{i \in T}$ and we write $\mathfrak{A}(\Delta)$ for the class of all algebras of type $\Delta$, where an algebra of type $\Delta$ is a pair $A = (A, (f_{i})_{i \in T})$, $f_{i}$ being an $n_{i}$-ary operation on the carrier set $A$ for each $i \in T$.

An equation $p = q$ ($p$ and $q$ can be considered as elements of the free algebra in $\mathfrak{A}(\Delta)$ generated by a set $\{x_{i} \mid i \in N\}$ of "variables") is called regular if $p$ contains the same variables as $q$.

If $\mathfrak{A} \subseteq \mathfrak{A}(\Delta)$, we write $\text{Eq}(\mathfrak{A})$ for the set of all equations and $\text{Eq}^{\text{reg}}(\mathfrak{A})$ for the set of all regular equations which are valid in each algebra in $\mathfrak{A}$. For a set $G$ of equations $\text{Md}(G)$ is the class consisting of all algebras in $\mathfrak{A}(\Delta)$ in which every equation of $G$ is valid.

It is clear that $\text{MdEq}(\mathfrak{A})$ is the smallest class in $\mathfrak{A}(\Delta)$ which contains $\mathfrak{A}$ and is definable by equations and that $\text{MdEq}^{\text{reg}}(\mathfrak{A})$ is the smallest class in $\mathfrak{A}(\Delta)$ which contains $\mathfrak{A}$ and is definable by regular equations ($\text{MdEq}$ and $\text{MdEq}^{\text{reg}}$ are closure operators on $\mathfrak{A}(\Delta)$).

By a well known theorem of G. Birkhoff, $\text{MdEq}(\mathfrak{A})$ is equal to $\mathfrak{HISP}(\mathfrak{A})$, where $\mathfrak{H}$, $\mathfrak{I}$, and $\mathfrak{P}$ are the operators defined by forming all homomorphic
images, all subalgebras, and all products of a given class of algebras (1). The purpose of this paper is to state an analogous result for $\text{MdEq}^{\text{reg}}$.

3. For $A = (A, (f_i)_{i \in \mathbb{N}})$ we define the one-point extension $\hat{A} = (\hat{A}, (\hat{f}_i)_{i \in \mathbb{N}})$ of $A$ by

$$\hat{A} := A \cup \{0\} \quad \text{(disjoint union)},$$

$$\hat{f}_i(a_1, \ldots, a_n) = \begin{cases} f_i(a_1, \ldots, a_n) & \text{if } a_1, \ldots, a_n \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathfrak{A} \subseteq \mathfrak{A}(A)$ let $\mathfrak{s}(\mathfrak{A})$ be the class $\{\hat{A} \mid A \in \mathfrak{A}\}$.

Now we can prove the following

**Lemma 1.** Let $A$ be a non-empty algebra in $\mathfrak{A}(A)$. Then $\text{Eq}^{\text{reg}}(A) = \text{Eq}(\hat{A})$.

**Proof.** Let $p(x_1, \ldots, x_m) = q(x_1, \ldots, x_m) \in \text{Eq}^{\text{reg}}(A)$. Then for all $a_1, \ldots, a_m \in A$ we have $p(a_1, \ldots, a_m) = q(a_1, \ldots, a_m)$. For $a_1, \ldots, a_m \in \hat{A}$, where at least one $a_i$ is equal to 0, we get $p(a_1, \ldots, a_m) = 0 = q(a_1, \ldots, a_m)$. Therefore $p = q \in \text{Eq}(\hat{A})$. Now let $p(x_1, \ldots, x_n) = q(y_1, \ldots, y_m)$ be an equation which is not regular. So assume without loss of generality $x_1 \notin \{y_1, \ldots, y_m\}$. Choosing an arbitrary $a \in A$ we get $p(0, a, \ldots, a) = 0$, $q(a, \ldots, a) \in A$ and, therefore, it follows $p = q \in \text{Eq}(\hat{A})$. Since $\text{Eq}(\hat{A}) \subseteq \text{Eq}(A)$, the proof is completed.

**Corollary 1.** If $\mathfrak{A} \subseteq \mathfrak{A}(A)$ and $\mathfrak{l}$ is the one-element algebra in $\mathfrak{A}(A)$, then $\text{Eq}^{\text{reg}}(\mathfrak{A}) = \text{Eq}(\mathfrak{s}(\mathfrak{A} \cup \{\mathfrak{l}\}))$.

**Proof.** We have

$$\text{Eq}^{\text{reg}}(\mathfrak{A}) = \bigcap_{A \in \mathfrak{A}} \text{Eq}^{\text{reg}}(A) = \bigcap_{A \in \mathfrak{A}} \text{Eq}^{\text{reg}}(A) \cap \text{Eq}^{\text{reg}}(\mathfrak{l})$$

$$= \bigcap_{A \in \mathfrak{A}} \text{Eq}(\hat{A}) \cap \text{Eq}(\hat{\mathfrak{l}}) = \bigcap_{A \in \mathfrak{A}} \text{Eq}(\hat{A}) \cap \text{Eq}(\hat{\mathfrak{l}}) = \text{Eq}(\mathfrak{s}(\mathfrak{A} \cup \{\mathfrak{l}\})).$$

As a simple consequence of this corollary we get

**Theorem 1.** Let $\mathfrak{A} \subseteq \mathfrak{A}(A)$. Then $\text{MdEq}^{\text{reg}}(\mathfrak{A}) = \mathcal{H} \mathcal{P} \mathcal{B} \mathcal{E}(\mathfrak{A} \cup \{\mathfrak{l}\})$.

**Proof.** $\text{MdEq}^{\text{reg}}(\mathfrak{A}) = \text{MdEq}(\mathfrak{s}(\mathfrak{A} \cup \{\mathfrak{l}\})) = \mathcal{H} \mathcal{P} \mathcal{B} \mathcal{E}(\mathfrak{A} \cup \{\mathfrak{l}\})$.

From this theorem there follows a simple criterion to decide whether an equational class can be defined by regular equations:

**Corollary 2 (2).** For an equational class $\mathfrak{A} \subseteq \mathfrak{A}(A)$ the following conditions are equivalent:

(1) For this result and other details not proved in this note see [1].

(2) It has been communicated to me that this result is due to B. Jónsson and E. Nelson.
(i) $\mathcal{A}$ can be defined by regular equations.

(ii) $\hat{\mathcal{A}}(\mathcal{A}) \subseteq \mathcal{A}$.

**Proof.** (i) is equivalent to $\text{MdEq}^{\text{reg}}(\mathcal{A}) \subseteq \mathcal{A}$, and because of $1 \in \mathcal{A}$ we have $\text{MdEq}^{\text{reg}}(\mathcal{A}) = \mathcal{HISP}(\mathcal{A})$. But from $\mathcal{HISP}(\mathcal{A}) \subseteq \mathcal{A}$ it follows $\hat{\mathcal{A}}(\mathcal{A}) \subseteq \mathcal{A}$, and from $\hat{\mathcal{A}}(\mathcal{A}) \subseteq \mathcal{A}$ it follows $\mathcal{HISP}(\mathcal{A}) \subseteq \mathcal{HISP}(\mathcal{A}) = \mathcal{A}$.

4. In the sequel all algebras are of type $\Delta = (n_t)_{t \in T}$ with $n_t \neq 0$ for all $t \in T$.

First we repeat the definition of a direct system of algebras and its sum (see [2]):

Let $(I, \leq)$ be a partially ordered set with the property that for any two elements $i, j \in I$ there is the least upper bound l.u.b.$(i, j)$ of $i$ and $j$.

A **direct system** of algebras from $\mathcal{A}(\Delta)$ over the set $(I, \leq)$ is a pair $((A_i)_{i \in I}, (\psi_{ij})_{i < j})$, where $A_i \in \mathcal{A}(\Delta)$ for all $i \in I$ and $\psi_{ij}: A_i \rightarrow A_j$ are homomorphisms for all $i, j \in I$ with $i \leq j$ such that

(a) $\psi_{ii} = \text{id}_{A_i}$ for all $i \in I$,

(b) $\psi_{ik} \circ \psi_{kj} = \psi_{ij}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

The **sum** of the direct system is an algebra $S = (S, (s_i)_{i \in I})$, where $S = \bigcup_{i \in I} A_i$ (disjoint union) and $s_i(a_1, \ldots, a_{n_i}) = f_i(\psi_{i_0 i}(a_1), \ldots, \psi_{i_0 i}(a_{n_i}))$,

where $a_j \in A_{i_j}$ for $1 \leq j \leq n_i$ and $i_0 = \text{l.u.b.}(i_1, \ldots, i_{n_i})$.

We write $\mathcal{P}_*(\mathcal{A})$ for the class of all sums of direct systems of algebras from $\mathcal{A}$.

**Lemma 2.** Let $\mathcal{A} \subseteq \mathcal{A}(\Delta)$. Then

(i) $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{P}_*(\mathcal{A})$,

(ii) $\mathcal{P}_*(\mathcal{A}) \subseteq \mathcal{P}(\mathcal{A})$.

**Proof.** (i) Let $(A_i)_{i \in I}$ be a family of algebras from $\mathcal{A}$. We consider $\prod_{i \in I} \hat{A}_i$ and for $J \subseteq I$ the sets $B_J = \{(a_i)_{i \in I} | a_i = O_i$ for $i \in J, a_i \in A_i$ for $i \notin J\}$.

It is easy to see that

$$\bigcup_{i \in I} \hat{A}_i = \bigcup_{J \subseteq I} B_J$$

and that $B_J$ defines a subalgebra $B_J$ of $\prod_{i \in I} \hat{A}_i$ which is isomorphic to $\prod_{i \in I \setminus J} \hat{A}_i$. Now realize that $(\mathcal{P}(I), \subseteq)$ is a partially ordered set with the l.u.b.-property and that $((\prod_{i \in I} \hat{A}_i)_{J \in \mathcal{P}(I)}, (\pi_{JJ'})_{J \subseteq J'})$ is a direct system over $\mathcal{P}(I)$, where $\pi_{JJ'}$ is the natural projection from $\prod_{i \in I \setminus J} \hat{A}_i$ to $\prod_{i \in I \setminus J'} \hat{A}_i$ whenever $J \subseteq J'$. Then it is easy to check that the sum of this direct system is
isomorphic to \( \prod_{i \in I} \hat{A}_i \), where the isomorphism is induced by the bijection from \( \bigcup_{J \subseteq I} B_J \) to \( \prod_{i \in I} \hat{A}_i \).

(ii) Let \( (A_i)_{i \in I}, (\psi_{ij})_{i \leq j} \) be a direct system of algebras from \( \mathfrak{A} \) and let \( S \) be its sum. Now consider for each \( k \in I \) the subalgebra \( B_k \) of \( \prod_{i \in I} \hat{A}_i \), where \( B_k = \{ (a_i)_{i \in I} \mid a_i = \psi_{ki}(a_k) \) for \( k \leq i \) and \( a_k \in A_k, a_i = 0_i \) otherwise\). Then \( B_k \cong A_k \) and these isomorphisms induce an obvious isomorphism between the subalgebra \( U \) of \( \prod_{i \in I} \hat{A}_i \) with \( U = \bigcup_{k \in I} B_k \) and \( S \). From this it follows \( S \in \mathcal{IPb}(\mathfrak{A}) \).

**Corollary 3.** \( \mathcal{IPb}(\mathfrak{A}) \) is a quasi-primitive class of algebras for any \( \mathfrak{A} \subseteq \mathfrak{A}(\Delta) \).

**Proof.** We have to show that \( \mathcal{IPb}(\mathfrak{A}) \) is closed with respect to the operators \( \mathcal{I} \) and \( \mathcal{P} \). First, \( \mathcal{IPb}(\mathfrak{A}) \subseteq \mathcal{IPb}(\mathfrak{A}) \) is a trivial statement. Now we consider \( \mathcal{IPb}(\mathfrak{A}) \). This class is contained in \( \mathcal{ISp}(\mathfrak{A}) \) and from Lemma 2, (ii), it follows \( \mathcal{IPb}(\mathfrak{A}) \subseteq \mathcal{IPb}(\mathfrak{A}) \subseteq \mathcal{IPb}(\mathfrak{A}) \). By (i) of Lemma 2 we get \( \mathcal{IPb}(\mathfrak{A}) \subseteq \mathcal{IPb}(\mathfrak{A}) \subseteq \mathcal{IPb}(\mathfrak{A}) \), which completes the proof.

Finally, we characterize the operator \( \text{MdEq}^{reg} \) for classes of algebras without nullary fundamental operations by

**Theorem 2.** Let \( \mathfrak{A} \subseteq \mathfrak{A}(\Delta) \). Then

\[
\text{MdEq}^{reg}(\mathfrak{A}) = \mathcal{IIPb}(\mathfrak{A}).
\]

**Proof.** By Lemma 2, (ii), we have \( \mathcal{IPb}(\mathfrak{A}) \subseteq \mathcal{IPb}(\mathfrak{A}) \) and, therefore, \( \mathcal{IIPb}(\mathfrak{A}) \subseteq \mathcal{IIPb}(\mathfrak{A}) \). Since a class defined by regular equations is closed with respect to the operators \( \mathcal{I}, \mathcal{P} \) (it is an equational class) and the operator \( \mathcal{E} \) (Lemma 1), \( \mathcal{IIPb}(\mathfrak{A}) \subseteq \text{MdEq}^{reg}(\mathfrak{A}) \), which proves the inclusion \( \mathcal{IIPb}(\mathfrak{A}) \subseteq \text{MdEq}^{reg}(\mathfrak{A}) \).

From Lemma 2, (i), it follows that \( \mathcal{IP}(\mathfrak{A} \cup \{1\}) \subseteq \mathcal{IP}(\mathfrak{A} \cup \{1\}) \)

\( = \mathcal{IP}(\mathfrak{A}) \), and by Theorem 1 we obtain \( \text{MdEq}^{reg}(\mathfrak{A} \cup \{1\}) \)

\( \subseteq \mathcal{IIPb}(\mathfrak{A}) \), which completes the proof of the theorem.

**Remark.** Note that Theorem 2 is proved without using the result (see [2]) that a class defined by regular equations is closed under formation of sums of direct systems, but using only the corresponding property of one-point extensions which are special cases of sums of direct systems. From this point of view Theorem 1 really shows that these special sums of direct systems are sufficient for characterizing the closure operator \( \text{MdEq}^{reg} \) while the connections between the formation of sums of general direct systems and one-point extensions are described in Lemma 2.
REFERENCES


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