A note on a multiplicative function

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An ordered set of integers \(a_1, \ldots, a_k\) is called a \(k\)-vector and is denoted by \(\{a_i\}\). The set of all \(k\)-vectors \(\{a_i\}\) where we take \(a_i(\text{mod } n)\) instead of \(a_i\), is said to constitute a complete residue system \((\text{mod } k, n)\). By the scalar multiple \(c\) of the vector \(\{a_i\}\) we mean the vector \(\{ca_i\}\). By the greatest common divisor (g.c.d.) of a vector \(\{a_i\}\) we mean the g.c.d. of the constituents and denote it by \(\langle a_i \rangle\). If \(d_1 = 1, \ldots, d_t = n\) are all the positive divisors of \(n\), then the complete residue system \((\text{mod } k, n)\) can be divided into \(t\) classes \(A(d_1), \ldots, A(d_t)\) such that the class \(A(d_j)\) \((j = 1, \ldots, t)\) contains all those vectors \(\{a_i\}\) of the complete residue system \((\text{mod } k, n)\) which are such that \(\langle a_i \rangle, n = n/d_j\). The set of elements \(A(n)\) is said to constitute a reduced residue system \((\text{mod } k, n)\).

The object of this note is to study certain properties of a sum associated with the arithmetic function \(f(m, n)\) whose values belong to the complex field and which has the following additional properties:

(i) \(f(m, n) = f(m', n),\) whenever \(m = m'(\text{mod } n),\)

(ii) \(f(m, n)f(m', n') = f(mn' + m'n, nn').\)

In particular, if \(d^{(k)}(m, n)\) is defined as \(\sum f(ms, n)\), the summation being over all \(s = s_1 + \ldots + s_k\), where \(\{s_i\}\) runs over all the elements of \(A(n)\), then we have:

**Theorem.**

\[
d^{(k)}(m, n) = \frac{J_k(n)}{J_k(n/g)} \mu(g, n)
\]

(Here \(J_k(n)\) is the Jordan totient which is the number of elements of \(A(n)\) and \(\mu(m, n)\) is defined as \(\sum f(r, n)\), the summation being over all \(r = r_1 + \ldots + r_k\) with \(\{r_i\}\) running over the elements of \(A(n/g)\), \(g\) being the g.c.d. of \(m, n\).)

For related literature on the function \(f(m, n)\) with side conditions, reference may be made to the author [3], [4] and Venkataraman [5].

**Lemma 1.** If \((n, n') = 1\) and the vector \(\{a_i\}\) ranges over the class \(A(d)\) \((\text{mod } k, n)\) and the vector \(\{a'_i\}\) ranges over the class \(A(d')\) \((\text{mod } k, n')\), then the vector \(\{a_in' + a'_in\}\) generates the class \(A(dd')\) \((\text{mod } k, nn')\).
Proof. If we set \( a = \{a_i\} \) and \( a' = \{a'_i\} \), then the set of all \( an' + a'n \) contains \( J_k(d)J_k(d') = J_k(dd') \) elements and the elements are distinct \((\text{mod} \ k, nn')\) for different \( a' \)'s and \( a' \)'s. Also \( an' + a'n \) belongs to the class \( A(dd') \ (\text{mod} \ k, nn') \). Therefore we have the lemma as stated above.

**Lemma 2.** \( \mu(m, n_1n_2) = \mu(m, n_1)\mu(m, n_2) \), whenever \( (n_1, n_2) = 1 \).

Proof. From the definition it follows that \( \mu(m, n) = \mu(g, n) \). Now the application of Lemma 1 to the definition of \( \mu(m, n) \) and the use of properties (i) and (ii) of \( f(m, n) \) yield the result.

**Lemma 3.** \( \mu(m_1m_2, n_1n_2) = \mu(m_1, n_1)\mu(m_2, n_2) \), whenever \( (m_1n_1, m_2n_2) = 1 \).

Proof. If \( (m_1, n_1) = g_1 \) and \( (m_2, n_2) = g_2 \), then \( (m_1m_2, n_1n_2) = g_1g_2 \). Now the left member of the equality in the lemma is equal to \( \mu(g_1g_2, n_1n_2) \), which by Lemma 2 is \( \mu(g_1g_2, n_1)\mu(g_1g_2, n_2) \), and this gives the result.

**Proof of the Theorem.** If \( d | n \), the \( J_k(n) \) elements of a reduced residue system \((\text{mod} \ k, n)\) can be decomposed into \( J_k(n)J_k(d) \) reduced residue systems \((\text{mod} \ k, d)\) (see Cohen [1], Lemma 7). Therefore

\[
\ell^{(k)}(m, n) = \frac{J_k(n)}{J_k(n/g)} \sum f(ms, n),
\]

where the summation is over all \( s = s_1 + \ldots + s_k \) and \( \{s_i\} \) runs over a reduced residue system \((\text{mod} \ k, n/g)\). Now put \( r = ms \); then \( m \{s_i\} \) runs over the class \( A(n/g) \ (\text{mod} \ k, n) \) as \( \{s_i\} \) ranges over the reduced residue system \((\text{mod} \ k, n/g)\). The rest follows from the definition of \( \mu(m, n) \).

**Remark 1.** The sum \( \ell^{(k)}(m, n) \) has the following multiplicative properties, which are immediate consequences of Lemmas 1 and 2 and the multiplicative property of the Jordan totient \( J_k(n) \):

\[
\ell^{(k)}(m, n_1n_2) = \ell^{(k)}(m, n_1)\ell^{(k)}(m, n_2), \quad \text{if} \quad (n_1, n_2) = 1,
\]

and

\[
\ell^{(k)}(m_1m_2, n_1n_2) = \ell^{(k)}(m_1, n_1)\ell^{(k)}(m_2, n_2) \quad \text{whenever} \quad (m_1n_1, m_2n_2) = 1.
\]

**Remark 2.** If \( f(m, n) = \exp(2\pi im/n) \), then \( \ell^{(k)}(m, n) \) reduces to the extension of the Ramanujan sum discussed in greater detail by Cohen in [1], § 3, and in [2].

**References**


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