LATTICE ORDERED GROUPS OF FINITE BREADTH

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The purpose of this note is to give the solution to a problem by Birkhoff (cf. [1], Problem 121) concerning lattice ordered groups (l-groups) of finite breadth (see Section 2). Let $G$ be an l-group, and consider the following conditions on $G$:

(a) There are elements $x, y \in G$ such that $x < y$ and $2x < 2y$.

(b) There are elements $x, y \in G$ such that $x \neq y$ and $2x = 2y$.

There exist l-groups $G$ satisfying (a) and (b) (cf. [1], p. 291, Example 5); Birkhoff asks if the pathological behaviour defined by conditions (a) and (b) can occur in an l-group of finite breadth.

We show that the answer is positive; for each positive integer $n > 1$ there is an l-group $G$ such that the breadth of $G$ equals $n$ and $G$ fulfils (a) and (b). It is not hard to verify that an l-group $G$ has finite breadth $n$ if and only if there exists a disjoint subset $S \subseteq G$ with $\text{card} S = n$ and if no disjoint subset of $G$ contains more than $n$ elements. Such l-groups were studied by Conrad and Clifford [4] (for $n = 2$), Conrad [2], Kokorin and Hisamiev [6] and Kokorin and Kozlov [7]. In [2] (cf. also [3], where a more general situation was dealt with) it was proved that any such l-group $G$ is a small lexicographic sum of linearly ordered groups $A_i$ ($i = 1, \ldots, n$), where $\{A_i\}_{i=1}^n = \mathcal{P}$ is the system of all maximal linearly ordered subgroups of $G$. We show that, for a lattice ordered group of finite breadth, condition (a) is equivalent with any one of the following conditions:

(c) There exists $A_i \in \mathcal{P}$ such that $A_i$ is not normal in $G$.

(d) There exists $0 < a \in G$ such that the interval $[0, a]$ is a chain and $a$ is disjoint with some of its conjugates.

1. Preliminaries. For the standard concepts concerning lattices and lattice ordered groups cf. [1] and [5]. We recall the following notions (cf. [3]):

Let $G = (G; \land, \lor, +)$ be an l-group, $a, b \in G$, $a \leq b$. The interval $[a, b]$ is the set $\{x \in G: a \leq x \leq b\}$. A subset $X \subseteq G$ is convex if $[x_1, x_2] \subseteq X$
whenever \( x_1, x_2 \in X \) and \( x_1 \leq x_2 \). A subset \( Y \subseteq G \) is called disjoint if \( y > 0 \) for each \( y \in Y \) and \( y_1 \wedge y_2 = 0 \) for any pair of distinct elements \( y_1, y_2 \in Y \). A system \( \mathcal{S} \) of convex \( l \)-subgroups of \( G \) is said to be disjoint if for any two distinct \( l \)-subgroups \( A_1, A_2 \in \mathcal{S} \) and any \( a_1 \in A_1, a_2 \in A_2 \) we have \( |a_1| \wedge |a_2| = 0 \). A disjoint system \( \mathcal{S} \) is said to be maximal if it is not a proper subset of a disjoint system of convex \( l \)-subgroups of \( G \). Let \( Y \) be a convex \( l \)-subgroup of \( G \) and \( x \in G \). If \( |x| \wedge |y| = 0 \) for each \( y \in Y \), we write \( Y \triangleleft x \).

Let \( X_1, \ldots, X_n \) be convex \( l \)-subgroups of \( G \) such that the group \((G; +)\) is the direct sum of \( X_i \) (\( i = 1, \ldots, n \)) and, for any \( x_i \in X_i \), \( x_1 + \cdots + x_n \geq 0 \) if and only if \( x_i \geq 0 \) for \( i = 1, \ldots, n \). Then the \( l \)-group \( G \) is said to be an \( l \)-direct sum of its \( l \)-subgroups \( X_i \), and we write \( G = X_1 \oplus \cdots \oplus X_n \).

More generally, let \( \mathcal{S} = \{X_i\} (i \in I) \) be a system of convex \( l \)-subgroups of \( G \) such that the group \((G; +)\) is the discrete direct sum of groups \((X_i; +)\) (\( i \in I \)). Suppose that, for any finite subset \( \{i_1, \ldots, i_n\} \subseteq I \) and any \( x_{i_k} \in X_{i_k} \), the relation \( x_{i_1} + \cdots + x_{i_n} \geq 0 \) implies \( x_{i_k} \geq 0 \) (\( k = 1, \ldots, n \)). Then \( G \) is the \( l \)-direct sum of the system \( \mathcal{S} \) and we then write \( G = \sum \oplus X_i (i \in I) \).

If \( i_1, \ldots, i_n \) are distinct elements of \( I \), \( x_i \in X_i \) and \( x = x_{i_1} + \cdots + x_{i_n} \), then we put \( x_{i_k} = x(X_{i_k}) \).

Now let \( I \) be a linearly ordered set and, for each \( i \in I \), let \( X_i \) be an \( l \)-group such that \( X_i \) is linearly ordered whenever \( i \) is not the least element of \( I \). Let \( H \) be the system of all mappings \( f : I \to \bigcup X_i \) with \( f(i) \in X_i \) for each \( i \in I \). For \( f \in H \) write \( I(f) = \{i \in I : f(i) \neq 0\} \). Let \( G \) be the system of all \( f \in H \) such that \( I(f) \) is well ordered. We define in \( G \) the operation \( + \) componentwise and we put \( f > 0 \) if \( I(f) \neq 0 \) and \( f(i_0) > 0 \), where \( i_0 \) is the least element of \( I(f) \). Then \( G \) is an \( l \)-group and it is called the lexicographic product of \( l \)-groups \( X_i \); we denote it by \( G = \Gamma X_i (i \in I) \).

Let \( A \) be an \( l \)-ideal of \( G \) such that \( g > a \) for any \( a \in A \) and \( g \in G^+ \setminus A \). Then \( G \) is a lexicographic extension of \( A \) and we then write \( G = \langle A \rangle \). A lexicographic extension \( G = \langle A \rangle \) is non-trivial if \( G \neq A \).

Let \( \mathcal{S}_1, \mathcal{S}_2, \ldots \) be systems of non-zero convex \( l \)-subgroups of \( G \) and \( K = \{1, 2, \ldots \} \). For any \( \mu \in K \) let \( A_\mu \) be the convex \( l \)-subgroup of \( G \) that is generated by the set \( \bigcup A_\mu \), where \( \mathcal{S}_\mu = \{A_\mu \} (i \in I_\mu) \). Assume that the following conditions are fulfilled:

(i) The system \( \mathcal{S}_1 \) is maximal disjoint.

(ii) If \( \mu \) is a positive integer, \( 1 < \mu \) and \( i \in I_\mu \), then either \( A_i^\mu \) equals \( A_{i-1}^\mu \) for some \( i_1 \in I_{\mu-1} \) or there is a convex \( l \)-subgroup \( B \) of \( G \) and a finite subset \( \{j_1, \ldots, j_m\} \subseteq I_{\mu-1} \), \( m > 1 \) such that \( B = A_{j_1}^{\mu-1} \oplus \cdots \oplus A_{j_m}^{\mu-1} \) and \( A_i^\mu \) is a non-trivial lexicographic extension of \( B \).

(iii) \( A_\mu = \sum \oplus A_i^\mu (i \in I_\mu) \) and \( A_\mu \) is an \( l \)-ideal of \( G \) for \( \mu = 1, 2, \ldots \).

(iv) \( G = \bigcup A_\mu (\mu \in K) \).
Then $G$ is said to be a small lexicographic sum of $l$-groups of the system $\mathcal{S}_1$.

If elements $x, y \in G$ are incomparable, we write $x \not| y$.

2. Breadth of a lattice. Let $L$ be a lattice. Suppose that $b = bL$ is the least positive integer such that any meet $x_1 \wedge \ldots \wedge x_n$ ($n > b$) is always a meet of $b$ of the $x_i$ (cf. [1], p. 99). Then $bL$ is the breadth of the lattice $L$; the lattice $L$ is said to be of finite breadth if $bL$ does exist.

A subset $X = \{x_1, \ldots, x_m\} \subseteq L$ will be called irreducible if $\inf(X \setminus \{x_i\}) > \inf X$ for each $i \in \{1, \ldots, m\}$. Then $bL = n > 1$ if and only if $L$ contains an irreducible subset with $n$ elements and if no subset $Y \subseteq L$ with $\card Y > n$ is irreducible.

Assertions 2.1-2.3 are easy to verify (cf. also [1], p. 32, Example 6).

2.1. $bL = 1$ if and only if $L$ is a chain.

2.2. If $bL$ exists and $L_1$ is a sublattice of $L$, then $bL_1$ exists and $bL_1 \leq bL$.

2.3. If $L$ is a direct product of lattices $L_1$ and $L_2$ with $\card L_1 > 1$ ($i = 1, 2$) and if $bL_1$ and $bL_2$ exists, then $bL = bL_1 + bL_2$.

2.4. Let $X$ be a disjoint subset of an $l$-group $G$, $\card X = n$, and let $H$ be the convex $l$-subgroup of $G$ generated by $X$. Assume that each interval $[0, x]$ ($x \in X$) is a chain. Then $bH = n$.

Proof. Let $X = \{x_1, \ldots, x_n\}$. For any $x_i$ there exists a maximal linearly ordered subgroup $X_i$ of $G$ containing $x_i$ and the system $\{X_i\}$ ($i = 1, \ldots, n$) is disjoint. Thus, from [2], Theorem 2, we infer that $H = X_1 \oplus \ldots \oplus X_n$; now, according to 2.1 and 2.3, we have $bH = n$, since the lattice $H$ is isomorphic to a direct product of lattices $X_i$ ($i = 1, \ldots, n$).

Let $G = \langle H \rangle$ and $x_1, \ldots, x_n \in G, x_1 \wedge \ldots \wedge x_n = x$. Since $G/H$ is linearly ordered, the set $\{x_1 + H, \ldots, x_n + H\}$ has the least element $x_k + H$ and $x_k < x_i$ for each $x_i \not| x_k + H$; because $x_k + H$ is a sublattice of $G$, we have $x \in x_k + H$ and $x = y_1 \wedge \ldots \wedge y_m$, where $\{y_1, \ldots, y_m\} = \{x_i : x_i \not| x_k + H\}$.

2.5. Let $G = \langle H \rangle$, $bH = n$. Then $bG = n$.

Proof. If $bG$ does exist, then $bG \geq n$ by 2.2. Let $m \geq n, x_i \in G$ ($i = 1, \ldots, m$), $x_1 \wedge x_2 \wedge \ldots \wedge x_m = x$ and assume that $X = \{x_1, \ldots, x_m\}$ is an irreducible subset of $G$. There is a subset $Y \subseteq X$ such that $Y \subseteq H + x$ and $\inf Y = \inf X$. Since $X$ is irreducible, we have $X = Y$. Write $x - x = z_i$. Then $\{z_1, \ldots, z_m\}$ is an irreducible subset of $H$, hence $m \leq n$. This shows that $bG = n$.

2.6. If $G$ is a small lexicographic sum of a system $\mathcal{S}_1 = \{B_1, \ldots, B_n\}$ of non-zero linearly ordered groups, then $bG = n$.

Proof. Assume that $G$ is a small lexicographic sum of a finite system $\mathcal{S}_1 = \{B_1, \ldots, B_n\}$ of non-zero linearly ordered groups. Then there is
a positive integer \( k \leq n \) such that (the notation is as in Section 1) \( G = A_k \). From 2.1, 2.3 and 2.5 we infer (by induction) that \( bA_m = n \) for \( m = 1, \ldots, k \). Thus \( bG = n \).

Consider the following example (cf. [1], p. 216, Example 6):

Let \( N \) be the set of all integers and let \( G \) be the set of all triples \( (x, y, z) \) with \( x, y, z \in N \). We define the operation \( + \) in \( G \) by the rule

\[
(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2),
\]

where \( y_3 = y_4 + y_5 \) and \( z_3 = z_1 + z_2 \) if \( x_2 \) is even, \( y_3 = z_1 + y_2 \) and \( z_3 = y_1 + +z_2 \) if \( x_2 \) is odd.

Further, we put \( (x, y, z) \geq 0 \) if either \( x > 0 \) or \( x = 0 \) and \( y \geq 0 \), \( z \geq 0 \). Then \( G \) is a lattice ordered group. Write \( Y = \{(0, y, 0) : y \in N \} \) and \( Z = \{(0, 0, z) : z \in N \} \). The \( l \)-group \( G \) is a small lexicographic sum of \( l \)-groups \( Y, Z \) and these are linearly ordered. Therefore, according to 2.6, we have \( bG = 2 \).

Put \( a = (1, 0, 0) \) and \( b = (1, 1, -1) \). Then \( a \neq b \) and \( 2a = (2, 0, 0) = 2b \), thus \( G \) fulfills (b). Further, write \( c = (1, 2, -1) \). The elements \( a, c \) are incomparable and \( 2c = (2, 1, 1) \geq 2a \), hence \( G \) satisfies (a). Let \( n \) be a positive integer, \( n - 2 = k > 0 \). Let \( B \) be the direct sum of \( k \) copies of \( N, H = G \oplus B \). Then, \( bH = n \) and \( H \) fulfills (a) and (b). Therefore, we have

**2.7. For any positive integer \( n \geq 2 \) there is a lattice ordered group \( G \) fulfilling (a) and (b) such that \( bG = n \).**

**2.8. Let \( G \) be an \( l \)-group. The following conditions are equivalent:**

(i) \( bG = n \);
(ii) \( G \) contains a disjoint subset with \( n \) elements and it does not contain any disjoint subset with more than \( n \) elements.

**Proof.** Assume that (i) holds and let \( X = \{x_1, \ldots, x_m\} \) be a disjoint set. Let \( H \) be the \( l \)-subgroup of \( G \) generated by \( X \). Then \( H \cap [0, x_1] = \{0, x_1\} \). According to 2.4, we have \( bH = m \), whence \( bG \geq m \), and this implies \( m \leq n \). Thus there exists a disjoint subset \( X \) of \( G \) with the greatest cardinality \( m_0 \leq n \). From [2], Theorem 1, it follows that \( G \) is a small lexicographic sum of \( m_0 \) non-zero linearly ordered groups and hence, by 2.6, \( bG = m_0 \). Thus \( n = m_0 \), and so (ii) is satisfied. Conversely, let (ii) hold. By [2] and 2.6, we obtain \( bG = n \).

**3. Condition (F).** Let us consider the following condition on \( G \neq \{0\} \):

(F) Any bounded disjoint subset of \( G \) is finite.

**3.1 (cf. [3], Theorem 6.1).** \( G \) fulfills (F) if and only if it is a small lexicographic sum of linearly ordered groups.

From the proof of this theorem that may be found in [3] it follows that if \( G \) satisfies (F), then it is a small lexicographic sum of the system
$\mathcal{S}_1$ consisting of all maximal non-zero linearly ordered subgroups of $G$. According to 2.8, any $l$-group $G$ of finite breadth satisfies (F). If $G$ satisfies (F), then it need not be of finite breadth.

Assume that $G$ satisfies (F). Let $\mathcal{S}_1$ be as above and let $\mathcal{S}_n (n = 2, 3, \ldots)$ be as in Section 1.

3.2. Assume that each $l$-group $A^1_i (i \in I_1)$ is an $l$-ideal of $G$. Then, for each $n > 1$ and each $i \in I_n$, the $l$-group $A^n_i$ is an $l$-ideal of $G$.

Proof. Assume that the assertion is valid for $n-1$, where $n > 1$, and let $i \in I_n$. There is $i_1 \in I_{n-1}$ such that $A^{n-1}_{i_1} \subset A^n_i$. Thus, for any $x \in G$, we have $A^{n-1}_{i_1} = x + A^{n-1}_{i_1} - x \subset x + A^n_i - x$. Moreover, from the construction of $\mathcal{S}_n$ devised in [3], it follows that there is $i_2 \in I_n$ such that $x + A^n_i - x = A^n_{i_2}$ and any two distinct elements of $\mathcal{S}_n$ are disjoint. Since $A^{n-1}_{i_1} \subset A^n_i \cap A^n_{i_2}$, we obtain $A^n_{i_2} = A^n_i$.

3.3. Assume that there is $A^1_{i_0} \in \mathcal{S}_1$ such that $A^1_{i_0}$ is not an $l$-ideal of $G$ and the number of $l$-groups that are conjugate to $A^1_{i_0}$ is finite. Then $G$ fulfills (a).

Proof. According to the assumption, there is $x \in G$ such that $-x + A^1_{i_0} + x \neq A^1_{i_0}$. Consider the mapping $\varphi: g \mapsto -x + g + x$ ($g \in G$). Each of the $l$-groups

\begin{equation}
\varphi(A^1_{i_0}), \varphi^2(A^1_{i_0}), \ldots, \varphi^n(A^1_{i_0}), \ldots
\end{equation}

is conjugate to $A^1_{i_0}$, hence the sequence (1) is finite and since $\varphi$ is an automorphism on $G$, there is the least positive integer $n > 1$ such that $\varphi^n(A^1_{i_0}) = A^1_{i_0}$. Then $A^1_{i_0}, \varphi(A^1_{i_0}), \ldots, \varphi^{n-1}(A^1_{i_0})$ are distinct $l$-groups. Choose $0 < z \in A^1_{i_0}$ and write

\[ y = x + z - \varphi(z) + \varphi^2(z) - \varphi^3(z) + \ldots + (-1)^{n-1} \varphi^{n-1}(z), \quad y_1 = -x + y. \]

Since $\varphi^k$ ($k = 1, 2, \ldots$) is an automorphism on the $l$-group $G$, each $l$-group $\varphi^k(A^1_{i_0})$ is a maximal linearly ordered subgroup of $G$, hence belongs to $\mathcal{S}_1$. Therefore, the system

\[ \mathcal{S}_0 = \{ A^1_{i_0}, \varphi(A^1_{i_0}), \ldots, \varphi^{n-1}(A^1_{i_0}) \} \]

is disjoint and so, according to Theorem 2 of [2], the convex $l$-subgroup $H$ of $G$ generated by the subgroups belonging to $\mathcal{S}_0$ is the $l$-direct sum of $l$-groups $A^1_{i_0}, \ldots, \varphi^{n-1}(A^1_{i_0})$. We have $y_1 \in A^1_{i_0} \oplus \varphi(A^1_{i_0}) \oplus \ldots \oplus \varphi^{n-1}(A^1_{i_0})$ and $\varphi^k(z) > 0$ ($k = 0, \ldots, n-1$); thus the element $y_1$ is incomparable with 0 and, therefore, $y_1 \not\sim x$. Further, we have

\[ \varphi(y_1) = \varphi(z) - \varphi^2(z) + \varphi^3(z) - \ldots + (-1)^{n-1} \varphi^n(z), \]

whence

\[ 2y = 2x + \varphi(y_1) + y_1 = 2x + (-1)^{n-1} \varphi^n(z) + z. \]

Now, we distinguish two cases:
(i) Suppose that \(( -1 )^{n - 1} \varphi^n(x) + z \neq 0\). Since \(\varphi^n(A^1_{0}) = A^1_{0}\), we have \(( -1 )^{n - 1} \varphi^n(x) + z \in A^1_{0}\) and because \(A^1_{0}\) is linearly ordered, we infer that the elements \(2x\) and \(2y\) are comparable and distinct; therefore, (a) is valid.

(ii) Assume that \(( -1 )^{n - 1} \varphi^n(x) + z = 0\). Write \(y_2 = z + y_1, y' = x + y_2\). Then we get \(y' = x + y_2\), and

\[
2y' = 2x + \varphi(y_2) + y_2 = 2x + \varphi(x) + \varphi(y_1) + z + y_1
\]

\[
= 2x + \varphi(x) + (( -1 )^{n - 1} \varphi^n(x) + z) + z = 2x + \varphi(x) + z > 2x,
\]

whence (a) holds.

As a corollary to 3.3, 2.8, and 3.1 we obtain

3.4. Let \(G\) be an \(l\)-group of finite breadth and assume that there exists \(A^1_{i} \in \mathcal{J}_1\) such that \(A^1_{i}\) is not normal in \(G\). Then \(G\) fulfills (a).

Assume that \(G\) satisfies (F) and that each \(A^1_{i}(i \in I_1)\) is normal. For \(0 < x \in G\), let \(I(x) = \{ i \in I_1: 0 < a \leq x \text{ for some } a \in A^1_{i} \}\). Then, for \(x, y \in G\), we have \(x \wedge y = 0\) if and only if \(I(x) \cap I(y) = \emptyset\).

Proof. Let \(x \wedge y = 0\). Assume that \(i \in I(x) \cap I(y)\). There exist \(a_1, a_2 \in A^1_{i}\) with \(0 < a_1 \leq x, 0 < a_2 \leq y\); because \(A^1_{i}\) is linearly ordered, we have \(0 < a_1 \wedge a_2 \leq x \wedge y = 0\), a contradiction. Conversely, let \(I_1(x) \cap I_1(y) = \emptyset\). If \(x \wedge y = z > 0\), then (since the system \(\mathcal{J}_1\) is maximal disjoint) there is \(i \in I_1\) and \(0 < a \in A^1_{i}\) with \(a \leq z\). From this we obtain \(i \in I_1(x) \cap I_1(y)\), a contradiction.

Let \(\varphi\) have the same meaning as in 3.3.

3.6. Assume that \(G\) satisfies (F) and that each \(A^1_{i}(i \in I_1)\) is normal. Let \(a, b \in G\), \(a \wedge b = 0\). Then \(a \wedge \varphi(b) = 0\).

Proof. Suppose to the contrary that \(a \wedge \varphi(b) = c > 0\). Then \(I(c) \neq \emptyset\); let \(i \in I(c)\). Thus \(i \in I(a)\) and \(i \in I(\varphi(b))\). Therefore, \(A^1_{i}\) non \(\delta\varphi(b)\), whence \(\varphi^{-1}(A^1_{i})\) non \(\delta b\). But \(\varphi^{-1}(A^1_{i}) = A^1_{i}\) and so \(A^1_{i}\) non \(\delta b\), and this implies \(i \in I(b)\). We get \(i \in I(a) \cap I(b)\); thus, according to 3.5, \(a \wedge b \neq 0\), a contradiction.

3.7. Assume that \(G\) satisfies (F) and each \(A^1_{i} \in \mathcal{J}_1\) is normal. Then \(G\) fulfills neither (a) nor (b).

Proof. Suppose that there are elements \(x, y \in G\) with \(x \not| y, 2x \leq 2y\). There is a positive integer \(n\) such that \(x, y, 2x, 2y \in A^1_{n}\); let \(n\) be the least positive integer with this property. Since \(A^1_{n}\) is discrete direct sum of \(l\)-groups \(A^1_{n}(i \in I_n)\), there must exist \(i_0 \in I_n\) such that \(x(A^1_{n}) | y(A^1_{n})\), \(2x(A^1_{n}) = 2y(A^1_{n})\). Write \(x(A^1_{n}) = x_1, y(A^1_{n}) = y_1\). It cannot occur that \(A^1_{n} \leq A^1_{n-1}\) for some \(i \in I_{n-1}\) because of the minimality of \(n\); moreover, \(n > 1\) since the \(l\)-groups \(A^1_{i}\) are linearly ordered. Thus \(A^1_{n} = \langle B \rangle\), \(A^1_{n} \neq B\) and \(B\) is a direct sum of two or more \(l\)-groups belonging to \(\mathcal{J}_{n-1}\). From \(x \not| y\) we obtain \(x + B = y + B\), whence \(z = -x_1 + y_1 \in B, z \not| 0\). There-
fore, \( z = z^+ - z^- \), \( z^+ > 0 \), \( z^- > 0 \) and \( z^+ \wedge z^- = 0 \). Hence, \( y_1 = x_1 + z \) and \( 2y_1 = 2x_1 + \varphi(z^+) - \varphi(z^-) + z^+ - z^- \).

According to 3.6, we have \( \varphi(z^+) \wedge z^- = 0 \), \( \varphi(z^-) \wedge z^+ = 0 \) and, clearly, \( \varphi(z^+) \wedge \varphi(z^-) = 0 \); therefore, \( (\varphi(z^+) + z^+) \wedge (\varphi(z^-) + z^-) = 0 \). From this it follows that \( \varphi(z^+) - \varphi(z^-) + z^+ - z^- \mid 0 \), whence \( 2y_1 \mid 2x_1 \), a contradiction.

3.8. Theorem. Assume that \( G \) satisfies \( (F) \) and that each \( A_i \in S \) has only a finite number of conjugates. Then condition (a) is equivalent with any one of the following conditions: (i) there exists \( A_i \in S \) that is not normal in \( G \); (ii) there exists \( 0 < a \in G \) such that \( [0, a] \) is a chain and the element \( a \) is disjoint with some of its conjugates. Moreover, if \( G \) satisfies (b), then it fulfills (a) as well.

Proof. The equivalence of conditions (a) and (i) follows from 3.3 and 3.7. Assume that (i) is valid and choose \( 0 < a \in A_i \). There is \( x \in G \) such that \(-x + A_i + x = A_i \in S_1 \), \( A_i \neq A_i \), hence \( A_i \wedge A_i = \{0\} \) and, therefore, \( a \wedge (-x + a + x) = 0 \). Since \( A_i \) is linearly ordered, \([0, a] \) is a chain. Conversely, let (ii) be satisfied. Because \([0, a] \) is a chain, there is \( A_i \in S_1 \) with \( a \in A_i \). If \( a \wedge (-x + a + x) = 0 \) for some \( x \in G \), then \(-x + a + x \notin A_i \), hence \(-x + A_i + x \neq A_i \). If (b) is valid, then, according to 3.7, (i) holds, and so (a) is satisfied.

As a corollary we obtain

3.8.1. Let \( G \) be an \( l \)-group of finite breadth. Then conditions (a), (i) and (ii) from 3.8 are equivalent. If \( G \) satisfies (b), then it fulfills (a) as well.

3.9. There exist \( l \)-groups of finite breadth satisfying (a) and not fulfilling (b).

Example. Let \( I \) be the set of all integers with the natural order and let \( X_i = N = I \) for each \( i \in I \), \( X = Y = \bigcup_{i \in I} X_i \). Let \( G \) be the set of all triples \((n, x, y)\) with \( n \in N \), \( x \in X \), \( y \in Y \). For any \( n \in N \) and \( x \in X \) let \( x^n \in X \) such that \( x^n(i) = x(i + n) \). Define the operation \(+ \) in \( G \) by the rule

\[
(n_1, x_1, y_1) + (n_2, x_2, y_2) = (n_1 + n_2, x_3, y_3),
\]

where \( x_3 = x_1^n + x_2, y_3 = y_1^n + y_2 \) for \( n \) even, and \( x_3 = y_1^n + x_2, y_3 = x_1^n + y_2 \) for \( n \) odd. \((G; +)\) is a group. Put \((n_1, x_1, y_1) \geq 0\) if either \( n_1 > 0 \) or \( n_1 = 0 \) and \( x_1 \geq 0, y_1 \geq 0 \). Then \( G \) is an \( l \)-group, \( bG = 2 \), and \( G \) fulfills (a) but not (b).

REFERENCES


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