ON MONOTONE UNIONS OF CLOSED n-CELLS

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1. Introduction. In [1] Fort showed that every open connected set in $E^n$ is a monotone union of closed $n$-cells. We extend this property to a larger class of spaces and study some properties of such monotone unions. All spaces are Hausdorff.

2. Definitions. A simplicial complex $K^n$ is monotonic if

$$K^n = \bigcup_{i=1}^{p} K_i,$$

where each $K_i$ is a subcomplex of $K^n$, $K_i$ is an $n$-simplex and $K_{i+1}$ $(1 \leq i \leq p-1)$ is obtained from $K_i$ by adding just one $n$-simplex to $K_i$ that has an $(n-1)$-simplex in common with $K_i$. If $F$ is a set, $|F|$ is its cardinality.


Theorem 1. If $K^n$ is a monotonic complex of dimension $n$ $(n \geq 2)$, then

$$K^n = \bigcup_{i=1}^{\infty} C_i,$$

where $C_i$ is a closed $n$-cell and $C_i \subseteq C_{i+1}$ for $i = 1, 2, 3, \ldots$

Proof. Let $K^n$ be monotonic so that

$$K^n = \bigcup_{i=1}^{p} K_i.$$

The proof is by induction on $p$. The inductive hypothesis we carry along is the following: For $p < a$ we have proved our result with the hypothesis that if $\{\sigma^{n-1}_j\}_{j=1}^{a}$ is any collection of $(n-1)$-simplices in $K^n$, then the union

$$\bigcup_{i=1}^{\infty} C_i = K^n$$
can be chosen so that \( \text{Bd} \, C_i \) meets each \( \sigma_i^{n-1} \) in an \((n-1)\)-dimensional proper set for all \( i \). For \( p = 1 \) the result follows. So assume it for \( p < a \). Let

\[
K^n = \bigcup_{i=1}^{a-1} K_i.
\]

\( \bigcup_{i=1}^{a-1} K_i \) is monotonic and has an \((n-1)\)-simplex in common with \( \sigma^n \), the \( n \)-simplex added to \( K_{a-1} \) to get \( K_a \). Call it \( \sigma^{n-1} \). By the inductive hypothesis we have \( K_{a-1} = \bigcup_{i=1}^{\infty} C_i \) and \( \text{Bd} \, C_i \cap \sigma^{n-1} \) has dimension \( n-1 \). There is on each \( \text{Bd} \, C_i \) a point \( x \) and a neighborhood of \( x \) in \( \text{Int} \, a^{n-1} \). To each \( C_i \) we attach a monotone union of closed \( n \)-cells in \( (\sigma^n - K_{a-1}) \cup A^{n-1} \) where \( A^{n-1} \) is an \((n-1)\)-simplex neighborhood of \( x \) in \( \text{Int} \, \sigma^{n-1} \). Call the resulting \( n \)-cells \( C_i \) and then \( K^n = \bigcup_{i=1}^{\infty} C_i \). The inductive hypothesis follows immediately.

By way of example consider a 3-book \( B^3 \); that is \( B^3 = T \times [0, 1] \), where \( T \) is a triod. By looking at Fig. 1 we can see the right monotone union since \( B^3 \) is a 1-1 continuous image of \( B_1 \).

![Fig. 1](image)

**4. Complexes of dimension** \( n \leq 1 \). A monotonic 0-complex is a point. So let \( K \) be a monotonic 1-complex. We ask when

\[
K = \bigcup_{i=1}^{\infty} I_i,
\]

where \( I_i = [a_i, b_i] \) is an arc with endpoints \( a_i, b_i \) while \( I_i \subset I_{i+1} \). The triod shows this is not always the case, while every connected 1-complex is monotonic. Without loss of generality let \( K = \bigcup_{i=1}^{\infty} I_i \), where \( a_i \rightarrow a \) and \( b_i \rightarrow b \). Then \( K \) is a very special continuous image of the closed interval \([0, 1]\) under \( f: [0, 1] \rightarrow K \) such that \( f|(0, 1) \) is a homeomorphism while \( f(0) = a, f(1) = b \), and so \( F = f^{-1}(a \cup b) \) contains at most four points.

If \( F \) contains four points then \( f \) identifies 0 or 1 with interior points \( c \) and \( d \) respectively. Then \( K \) is determined entirely by the order 0, \( c, d, 1 \). The two figures appear in Fig. 2.
If \( F \) contains 3 points, then \( f \) identifies a single interior point. The other figures appear in Fig. 2.

These figures represent the termination of a selfavoiding walk discussed in [2], if growth occurs at each end. Similar figures are found in the study of 1-1 maps as seen in [3].

\[
\begin{align*}
|F| = 4 & & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram1.png}
\end{array} \\
|F| = 3 & & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram2.png}
\end{array} \\
|F| = 2 & & \begin{array}{c}
\includegraphics[width=0.2\textwidth]{diagram3.png}
\end{array}
\end{align*}
\]

Fig. 2

5. **Products and 1-1 maps.** Suppose \( X \) is a topological space and \( X \) is a monotone union of closed \( n \)-cells. Then if \( Y \) is a 1-1 continuous image of \( X \), \( Y \) has the property as well. If \( X \) and \( Y \) both have the property, then \( X \times Y \) is a monotone union of closed cells. The converse of the last statement if false as shown by the product of a triod and an interval. From this remark and Theorem 1 we have.

6. **The monotone topology.** If \( X = \bigcup C_i \), where \( C_i \subset C_{i+1} \) is a closed \( n \)-cell, the monotone topology is obtained by defining a new topology on the set \( X \). If \( X \) is an interior point of some \( C_i \), then a basis at \( x \) is any basis at \( x \) in \( \text{Int} \, C_i \). If \( X \) is never in the interior of a cell we take as basis at \( x \) all sets of form \( \bigcup_{a_x} U_x^j \) where \( x \in C_{a_x} \) and \( U_x^j \) is a neighborhood of \( x \) in \( C_j \). Let \( \tilde{X} \) be the resulting space. Then the natural map \( \tilde{N}: \tilde{X} \rightarrow X \) is continuous. \( \tilde{X} \) has a weak topology.

The above results can be generalized to infinite complexes.
REFERENCES


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