ON THE BINDING NUMBER OF SOME HALLIAN GRAPHS

1. Definitions and notation. We consider only finite undirected graphs without loops or multiple edges. Our terminology and notation will be standard except for indicated. For a graph $G = (V, E)$ and a set $X \subseteq V$ we denote by $\Gamma_G(X)$ (or, briefly, $\Gamma(X)$) the set of vertices joined to vertices in $X$. A set of independent edges that cover all vertices of a graph is called a 1-factor of that graph. By a $(1, 2)$-factor of a graph $G$ we mean a set of independent edges or vertex disjoint cycles which cover all vertices of $G$. Obviously, we can restrict the cycles in the above definition to odd ones.

A graph $G$ is hallian if $|\Gamma(X)| \geq |X|$ for any set $X \subseteq V(G)$ or, equivalently, if $G$ has a $(1, 2)$-factor. Obviously, $G$ is a hallian graph if its vertices can be covered by a set of vertex disjoint even paths or odd cycles. A graph $G$ is $k$-hallian if for any set $A$ of vertices of order at most $k$ the subgraph of $G$ induced by the set $V(G) \setminus A$ is hallian. The largest $k$ such that $G$ is $k$-hallian is called the hallian index of $G$ and is denoted by $h(G)$. The vertex connectivity $\kappa(G)$ of a graph $G$ is the minimum number of vertices whose removal results in a disconnected or trivial graph. Clearly,

$$h(G) \leq \delta(G) - 1 \quad \text{and} \quad \kappa(G) \leq \delta(G),$$

where $\delta(G)$ denotes the minimum degree among the vertices of $G$.

For concepts not defined here, see [1].

2. Some properties of hallian graphs. The main purpose of this section is to show relations between orders of $\Gamma(X)$ and $X$, $X \subseteq V(G)$, for $k$-hallian and $l$-connected graphs. Let $G$ be a given graph and let

$$\mathcal{F}_G = \{X \subseteq V(G): X \neq \emptyset \text{ and } \Gamma(X) \neq V(G)\}.$$

**Lemma 1.** Let $G$ be a graph on $n$ vertices. If $G$ is $l$-connected, then

$$|\Gamma(X) \setminus X| \geq l$$

for any set $X \in \mathcal{F}_G$. 
Proof. For \( l = 0 \) the lemma is obvious. Let \( l \geq 1 \) and \( X \in \mathcal{F}_G \). There exists a vertex \( v \notin \Gamma(X) \). If \( v \in X \setminus \Gamma(X) \), then \( \Gamma(v) \) is disjoint from \( X \), and so
\[
|\Gamma(X) \setminus X| \geq |\Gamma(v)| \geq \delta(G) \geq l.
\]
If \( v \in V \setminus (X \cup \Gamma(X)) \), then \( \Gamma(X) \setminus X \) is a cutset of \( G \), so \( |\Gamma(X) \setminus X| \geq l \).

Theorem 1. Let \( G \) be an \( l \)-connected and \( k \)-hallian graph on \( n \) vertices. Then
\[
|\Gamma(X)| \geq |X| + r
\]
for any set \( X \in \mathcal{F}_G \), where \( r = \min \{l, k\} \).

Proof. Let \( X \in \mathcal{F}_G \). By Lemma 1 we have
\[
|\Gamma(X) \setminus X| \geq r,
\]
so choose \( A \subseteq \Gamma(X) \setminus X \) with \( |A| = r \). The graph \( G' = \langle V(G) \setminus A \rangle \) is \((k-r)\)-hallian, hence hallian, whence \( |\Gamma_{G'}(X)| \geq |X| \). Thus
\[
|\Gamma_G(X)| = |\Gamma_{G'}(X)| + |A| \geq |X| + r,
\]
as required.

Corollary 1. Let \( G \) be a hallian graph on \( n \) vertices and let
\[
r = \min \{h(G), \chi(G)\}.
\]
Then there exists a set \( X_0 \in \mathcal{F}_G \) such that
\[
|\Gamma(X_0)| = |X_0| + r.
\]

Proof. We consider two cases.

Case 1. \( r = h(G) \). Then there exists a set \( A \subseteq V(G) \), \( |A| = r + 1 \), such that the graph \( G' = \langle V(G) \setminus A \rangle \) is not hallian. Thus there exists \( X \subseteq V(G) \setminus A \) such that \( |\Gamma_{G'}(X)| < |X| \), and
\[
|\Gamma_G(X)| \leq |\Gamma_{G'}(X)| + |A| < |X| + r + 1.
\]
So \( |\Gamma(X)| \leq |X| + r \). Since \( \Gamma(X) \neq V(G) \), the converse inequality follows from Theorem 1, and the result is proved in this case.

Case 2. \( r = \chi(G) \). Let \( Y \subseteq V(G) \) be a cutset of \( r \) vertices and let \( X \) be the set of vertices on one component of \( G \setminus Y \). Then, evidently, \( |\Gamma(X)| \leq |X| + r \). Since the converse inequality follows again from Theorem 1, the result is also proved in this case.

Theorem 2. Let \( G \) be a graph on \( n \) vertices. If any set \( X \in \mathcal{F}_G \) satisfies
\[
|\Gamma(X)| \geq |X| + k,
\]
then \( G \) is \( k \)-hallian and \( k \)-connected.

Proof. For \( k = 0 \) the theorem is clear. Let \( k \geq 1 \). Suppose \( G \) is not \( k \)-connected or \( G \) is not \( k \)-hallian. Then \( \chi(G) < k \) or \( h(G) < k \). Let
\[
r = \min \{\chi(G), h(G)\}.
\]
By Corollary 1 there exists a set \( X_0 \in \mathcal{F}_G \) such that \( |\Gamma(X_0)| = |X_0| + r \), where \( r < k \), a contradiction.

**Corollary 2.** If \( k' \) is the largest integer for which the assertion of Theorem 2 holds, then \( h(G) = k' \) or \( \chi(G) = k' \).

3. **On the binding number of halian graphs.** The binding number of \( G \), denoted by \( \text{bind}(G) \), is defined by

\[
\text{bind}(G) = \min_{X \in \mathcal{F}_G} \frac{|\Gamma(X)|}{|X|}.
\]

The binding number was intensively investigated by Woodall; see [4] and its references. If \( \text{bind}(G) \) is large, then the vertices of \( G \) are well bound together, i.e., \( G \) has a lot of edges fairly well distributed. If the binding number is large, then the minimum degree, the connectivity, the chromatic number are also large. Clearly, \( \text{bind}(G) = 0 \) if and only if \( G \) has an isolated vertex.

Woodall has proved in [4] the following

**Proposition 1.** If \( G \) is a graph on \( n \) vertices, then

\[
\text{bind}(G) \leq \frac{n-1}{n-\delta(G)}.
\]

**Proposition 2.** \( \text{bind}(K_n) = n-1 \) for \( n \geq 1 \).

**Proposition 3.** If \( n \geq 3 \), then

\[
\text{bind}(C_n) = \begin{cases} 
1 & \text{if } n \text{ is even}, \\
\frac{(n-1)/(n-2)} & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proposition 4.** If \( n \geq 1 \), then

\[
\text{bind}(P_n) = \begin{cases} 
1 & \text{if } n \text{ is even}, \\
\frac{(n-1)/(n+1)} & \text{if } n \text{ is odd}.
\end{cases}
\]

We note that \( G \) is halian if and only if \( \text{bind}(G) \geq 1 \). Using Theorem 1 and the next lemma we shall obtain a lower bound for \( \text{bind}(G) \), where \( G \) is a halian graph.

**Lemma 2.** Let \( G \) be a graph on \( n \) vertices. If \( X \in \mathcal{F}_G \), then

\[
|X| \leq n-\delta(G).
\]

**Proof.** Let \( v \in V(G) \setminus \Gamma(X) \). Then \( X \cap \Gamma(v) = \emptyset \), and \( |\Gamma(v)| \geq \delta(G) \). The result is immediate.

**Theorem 3.** If a graph \( G \) on \( n \) vertices is \( k \)-halian and \( l \)-connected, and \( r = \min \{k, l\} \), then

\[
\text{bind}(G) \geq \frac{n-\delta(G)+r}{n-\delta(G)}.
\]
Proof. If \( X \in \mathcal{F}_G \), then \(|\Gamma(X)| \geq |X| + r\) by Theorem 1. Thus, by the above and Lemma 2,

\[
\frac{|\Gamma(X)|}{|X|} \geq \frac{|X| + r}{|X|} \geq \frac{n - \delta(G) + r}{n - \delta(G)}.
\]

The result follows.

**Lemma 3.** If a graph \( G \) on \( n \) vertices has \( h(G) = \delta(G) - 1 \) and \( x(G) \geq h(G) \), then

\[
\text{bind}(G) = \frac{n - 1}{n - \delta(G)}.
\]

**Proof.** As a consequence of Theorem 3 we have

\[
\text{bind}(G) \geq \frac{n - 1}{n - \delta(G)}
\]

and by Proposition 1 we obtain the required equality.

4. The binding number of the Cartesian product of some graphs. An application of the preceding results. Let \( G_1 \) and \( G_2 \) be graphs with vertex sets

\[
V(G_1) = \{x_1, \ldots, x_n\} \quad \text{and} \quad V(G_2) = \{y_1, \ldots, y_m\}.
\]

The Cartesian product of graphs \( G_1 \) and \( G_2 \) is a graph \( G_1 \times G_2 \) with vertex set \( V(G_1) \times V(G_2) \) and

\[
\{(x_i, y_j), (x_k, y_l)\} \in E(G_1 \times G_2)
\]

if and only if \( i = k \) and \( \{y_j, y_l\} \in E(G_2) \) or \( j = l \) and \( \{x_i, x_k\} \in E(G_1) \).

Kane et al. [2] have given the following

**Conjecture.** Let \( n \geq 3 \) and \( m \geq 3 \). Then

\[
\text{bind}(C_n \times C_m) = \frac{mn - 1}{mn - 4}
\]

if \( mn \) is odd.

Using Theorem 1 and Lemma 3 we will prove this Conjecture. Moreover, in a similar way we calculate \( \text{bind}(P_n \times C_m) \), \( \text{bind}(C_n \times K_m) \) and \( \text{bind}(P_n \times K_m) \).

The connectivity of the Cartesian product of two connected graphs is described by Sabidussi [3].

**Proposition 5 ([3]).** If \( G_1 \) is \( l_1 \)-connected and \( G_2 \) is \( l_2 \)-connected, then \( G_1 \times G_2 \) is \( (l_1 + l_2) \)-connected.

**Corollary 3.** The graph \( (C_n \times C_m) \) is 4-connected for \( n \geq 3 \), \( m \geq 3 \).

**Corollary 4.** The graph \( (P_n \times C_m) \) is 3-connected for \( n \geq 2 \), \( m \geq 3 \).

**Corollary 5.** The graph \( (C_n \times K_m) \) is \( (m + 1) \)-connected for \( n \geq 3 \), \( m \geq 3 \).
Corollary 6. The graph \((P_n \times K_m)\) is \(m\)-connected for \(n \geq 2, m \geq 3\).

Moreover, if \(G\) is \(l\)-connected and \(\delta(G) = l\), then \(\chi(G) = l\). Hence

\[
\chi(C_n \times C_m) = 4, \quad \chi(P_n \times C_m) = 3, \quad \chi(C_n \times K_m) = m + 1, \quad \chi(P_n \times K_m) = m.
\]

Lemma 4. If \(H\) is a spanning subgraph of \(G\) and \(H\) is \(k\)-hallian, then \(G\) is \(k\)-hallian. Moreover, \(h(H) \leq h(G)\).

Corollary 7. Let \(G_1\) and \(G_2\) be hallian graphs. Then

\[
h(G_1 \times G_2) \geq \max\{h(G_1), h(G_2)\}.
\]

Moreover, if either \(G_1\) or \(G_2\) (but not both) is hallian, then \(h(G_1 \times G_2)\) is greater than or equal to the hallian index of this graph.

Lemma 5. If \(n \geq 2, m \geq 3\), then

\[
h(P_n \times C_m) = \begin{cases} 2 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}
\]

Proof. Let \(G = P_n \times C_m\). Since

\[
h(C_m) = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even,} \end{cases}
\]

we have

\[
h(G) = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ 0 & \text{if } m \text{ is even.} \end{cases}
\]

If \(m\) is even, then removing any vertex \(x\) will destroy the hallian property since \(G - x\) has an odd number of vertices but no odd cycles. Thus \(h(G) = 0\).

Assume that \(m\) is odd. Let \(A \subseteq V(G)\) and \(|A| = 2\). If \(A\) consists of vertices of two copies of \(C_m\), then the vertices of \(G - A\) are covered by two even paths \(C_m - x\) and odd cycles, the remaining copies of \(C_m\). If \(A\) consists of vertices of the same copy \(C_m\), then it is not difficult to see that the vertices of \(C_m - A\) induce exactly one odd path and at most one even path. One of the end vertices of the odd path and the corresponding vertex of an adjacent copy \(C_m\) can be covered by an edge (see Fig. 1). In this way vertices of \(C_m - A\) and vertices of an adjacent copy \(C_m\) can be covered by even paths (Fig. 1), so \(G - A\) is again hallian. Thus \(h(G) \geq 2\). On the other hand, \(\delta(G) = 3\), so \(h(G) = 2\).

Fig. 1
Lemma 6. Let \( n \geq 3 \), \( m \geq 3 \). Then

\[
h(C_n \times C_m) = \begin{cases} 
3 & \text{if } m \text{ or } n \text{ is odd}, \\
0 & \text{if } m \text{ and } n \text{ are even}.
\end{cases}
\]

Proof. Let \( G = C_n \times C_m \). By Lemmas 4 and 5 we obtain

\[
h(G) \geq \begin{cases} 
2 & \text{if } m \text{ or } n \text{ is odd}, \\
0 & \text{if } m \text{ and } n \text{ are even}.
\end{cases}
\]

If \( m \) and \( n \) are both even, then the result follows by the argument of Lemma 5. So suppose that one of them, say \( m \), is odd. Let \( A \subseteq V(G) \) and \( |A| = 3 \). If \( A \) consists of vertices of three copies of \( C_m \), then it is clear that \( G - A \) has a cover by even paths \( C_m - A \) and odd cycles \( C_m \). If \( A \) consists of two vertices of the same copy \( C_m \) and one from another copy, then the vertices of \( G - A \) can be covered by even paths and odd cycles; see the proof of Lemma 5 (noting now that each copy of \( C_m \) has two others adjacent to it). If \( A \) consists of vertices of the same copy \( C_m \), then the vertices of \( C_m - A \) induce only even paths or exactly two odd paths and at most one even path. In the first possibility it is clear that the vertices of \( G - A \) are covered by even paths and odd cycles. In the second possibility the vertices of \( C_m - A \) and all vertices of the two copies adjacent to it can be covered by even paths (see Fig. 2) so that the graph \( G - A \) is still hollian. Thus \( h(G) \geq 3 \). But \( \delta(G) = 4 \), and so \( h(G) = 3 \).

![Fig. 2](image)

Theorem 4. If \( n \geq 3 \), \( m \geq 3 \), then

\[
\text{bind}(C_n \times C_m) = \begin{cases} 
\frac{mn-1}{2} & \text{if } m \text{ or } n \text{ is odd}, \\
\frac{mn-4}{2} & \text{if } m \text{ and } n \text{ are even}.
\end{cases}
\]

Proof. Let \( G = C_n \times C_m \). If \( m \) or \( n \) is odd, then by Corollary 3 and Lemma 6 the graph \( G \) satisfies the hypothesis of Lemma 3 and the result follows. If \( m \) and \( n \) are even, then by Lemma 6 and Corollary 1 we have \( h(G) = 0 \) and \( \chi(G) = 4 \). According to Corollary 1 there exists a set \( X_0 \in \mathcal{F}_0 \) such that \( |\Gamma(X_0)| = |X_0| \), so \( \text{bind}(G) = 1 \).
Using Lemmas 3 and 5 and Corollaries 1 and 4 we prove in the same way as Theorem 4 the following

**Theorem 5.** If \( n \geq 2, m \geq 3 \), then

\[
\text{bind}(P_n \times C_m) = \begin{cases} 
\frac{mn-1}{mn-3} & \text{if } m \text{ is odd,} \\
1 & \text{if } m \text{ is even.}
\end{cases}
\]

Now we will calculate \( \text{bind}(P_n \times K_m) \) and \( \text{bind}(C_n \times K_m) \). First we notice that the following lemma is true:

**Lemma 7.** If \( n \geq 2, m \geq 3 \), then \( h(P_n \times K_m) = m-1 \).

**Proof.** Let \( G = P_n \times K_m \). By Lemma 4 we have \( h(P_n \times K_m) \geq m-1 \). Let \( A \subseteq V(G) \) with \( |A| = m-1 \). Note that removing at most \( m-2 \) vertices from each copy of \( K_m \) leaves a hallian graph, since the remaining vertices of each \( K_m \) can be covered by a single edge or a cycle. If all vertices of \( A \) are of the same copy \( K_m \), then the remaining vertex can be paired by an edge with a vertex in an adjacent copy of \( K_m \), and it is easy to see that the graph \( G - A \) is hallian. Thus \( h(G) \geq m-1 \). Since \( \delta(G) = m \), it follows that \( h(G) = m-1 \).

**Lemma 8.** If \( n \geq 3, m \geq 3 \), then \( h(C_n \times K_m) = m \).

**Proof.** Let \( G = C_n \times K_m \) and suppose that at most \( m \) vertices of \( G \) are removed. The resulting graph is hallian by the argument of Lemma 7 unless all the vertices removed are from the same copy of \( K_m \), in which case the result is obvious. Thus \( h(G) \geq m \). But \( \delta(G) = m+1 \), and so \( h(G) = m \).

From Lemmas 7 and 3 we obtain

**Theorem 6.** If \( n \geq 2, m \geq 3 \), then

\[
\text{bind}(P_n \times K_m) = \frac{mn-1}{mn-n}.
\]

From Lemmas 8 and 3 we obtain

**Theorem 7.** If \( n \geq 3, m \geq 3 \), then

\[
\text{bind}(C_n \times K_m) = \frac{mn-1}{mn-n-1}.
\]

We express our sincerest thanks to the referee for his useful suggestions towards the improvement of this paper. He brought to our attention that the Conjecture of Kane et al. [2] has been also settled in the affirmative by D. R. Guichard (see Ars Combinatoria 19 (1985), pp. 175–178).

**References**


DEPARTMENT OF APPLIED MATHEMATICS
HIGHER COLLEGE OF ENGINEERING
ZIELONA GÓRA, POLAND

INSTITUTE OF MATHEMATICS
PEDAGOGICAL COLLEGE
ZIELONA GÓRA, POLAND