On balls and totally geodesic submanifolds

by P. G. WALCZAK (Łódź)

0. Totally geodesic submanifolds of a Riemannian manifold $M$ can be characterized as submanifolds $N$ such that any point $x$ of $N$ admits an open neighbourhood $U \subset N$ such that $d_M | U \times U = d_N | U \times U$, where $d_M$ and $d_N$ denote the distance functions on $M$ and $N$, respectively. In this note, we show that a compact submanifold $N$ of $M$ is totally geodesic if and only if there exists a positive number $\varepsilon > 0$ such that

$$B_N(x, r) = B_M(x, r) \cap N$$

for any $x$ of $N$ and $r$ of $(0; \varepsilon)$, where $B_M(x, r)$ and $B_N(x, r)$ denote the centred at $x$ open balls of radii $r$ on $M$ and $N$, respectively. (Recall, that the relation

$$B_N(x, r) \subset B_M(x, r) \cap N$$

holds for any submanifold $N$ of $M$, any $x$ and $r$.) If $N$ is a submanifold of $M$ and equality (1) holds for any $x$ of $N$ and $r$ of $(0; \varepsilon)$, then we say that $N$ is $\varepsilon$-regular. We denote by $\varepsilon(N)$ the smallest upper bound of the set of all $\varepsilon$ such that $N$ is $\varepsilon$-regular.

1. Theorem A. (a) If a submanifold $N$ of a Riemannian manifold $M$ is $\varepsilon$-regular for some $\varepsilon > 0$, then it is totally geodesic. (b) If $N$ is compact and totally geodesic, then it is $\varepsilon$-regular for some $\varepsilon > 0$.

Proof. (a) Suppose that $N$ is $\varepsilon$-regular. It is sufficient to show that if $x, y \in N$ and $d_N(x, y) < \varepsilon$, then $d_M(x, y) \geq d_N(x, y)$. In order to do this, let $a = d_N(x, y) < r < \varepsilon$. Then $y \in B_N(x, r) - B_N(x, a) = (B_M(x, r) \cap N) - (B_M(x, a) \cap N) = (B_M(x, r) - B_M(x, a)) \cap N$. It follows that $y \notin B_M(x, a)$, i.e. that $d_M(x, y) \geq a$.

(b) Suppose that $N$ is compact and totally geodesic. For any $x$ of $N$ denote by $g(x)$ the radius of injectivity of $M$ at $x$ ([2], § 5.2). The function $N \ni x \mapsto g(x)$ is continuous. Therefore, the number $\delta = \min_N g$ is positive. If $x \in N$, $v \in T_x N$, $|v| = 1$, $|t| < \delta$ and $y = \exp(tv)$, then $d_N(x, y) = d_M(x, y) = |t|$. In fact, if $a = d_N(x, y)$, then $a \leq L(c) = |t| < \delta$, where $c : \langle 0, 1 \rangle \to N$ is a
regular curve given by $c(s) = \exp(stv)$, and $y = \exp(aw)$ for some unit vector $w$ of $T_yN$. Since $|aw| < g(x)$, $|tv| < g(x)$, and $\exp(aw) = \exp(tv)$, we have $aw = tv$ and $a = |t|$. The similar argumentation shows that $d_M(x, y) = |t|$.

For any $x$ of $N$ put $\varepsilon_x = d_M(x, N-B_N(x, \frac{1}{2}\delta))$.

Let us take points $x$ and $y$ of $N$ such that $d_N(x, y) < \frac{1}{2}\delta$. There exists a point $z_0$ of the set $N-B_N(x, \frac{1}{2}\delta)$ such that $\varepsilon_x = d_M(x, z_0)$. Clearly,

$$d_N(y, z_0) \geq d_N(x, z_0) - d_N(x, y) \geq \frac{1}{2}\delta - d_N(x, y).$$

If $d_N(y, z_0) \geq \frac{1}{2}\delta$, then $z_0 \in N-B_N(y, \frac{1}{2}\delta)$ and

$$\varepsilon_y \leq d_M(y, z_0) \leq d_M(y, x) + d_M(x, z_0) \leq \varepsilon_x + d_N(x, y).$$

If $d_N(y, z_0) < \frac{1}{2}\delta$, then there exists an unit vector $v$ of $T_yN$ such that $z_0 = \exp(tv)$, $t = d_N(y, z_0)$. Put $z = \exp((t+d_N(x, y))v)$. Since $t + d_N(x, y) < \delta \leq g(y)$, we have $d_N(y, z) = t + d_N(x, y)$ and $d_N(z, z_0) = d_N(x, y)$.

Therefore,

$$\varepsilon_y \leq d_M(y, z) \leq d_M(y, x) + d_M(x, z_0) + d_M(z_0, z) \leq \varepsilon_x + 2d_N(x, y).$$

From (3) and (4) it follows that if $x, y \in N$ and $d_N(x, y) < \frac{1}{2}\delta$, then

$$|\varepsilon_x - \varepsilon_y| \leq 2d_N(x, y).$$

We conclude that the function $N \ni x \mapsto \varepsilon_x$ is continuous.

Put $\varepsilon = \min \{\varepsilon_x; x \in N\}$. We claim that $N$ is $\varepsilon$-regular. In fact, if $x \in N$, $r < \varepsilon$, and $y \in N \cap B_M(x, r)$, then $d_M(x, y) < \frac{1}{2}\delta$ (otherwise $y \in N-B_N(x, \frac{1}{2}\delta)$ and $d_M(x, y) \geq \varepsilon_x \geq \varepsilon > r$) and $d_N(x, y) = d_M(x, y) < r$.

This ends the proof.

In other words, Theorem A says that (a) if $\varepsilon(N) > 0$, then $N$ is totally geodesic and (b) if a submanifold $N$ is compact and totally geodesic, then $\varepsilon(N) > 0$.

**Examples.** If $S^k$ (resp. $P^kR$) is considered as a totally geodesic submanifold of $S^m$ (resp., of $P^mR$), $k < m$, then $\varepsilon(S^k) = \varepsilon(P^kR) = +\infty$. If $N$ is the totally geodesic submanifold of the torus $T = R^2/Z^2$ obtained by the projection of the line $L \subset R^2$ given by the equation

$$px_1 + qx_2 + c = 0,$$

then

(a) $\varepsilon(N) = 0$ when the number $p/q$ is irrational,

(b) $\varepsilon(N) = +\infty$ when $p, q \in Z$ and $p^2 + q^2 = 1$,

(c) $\varepsilon(N) = 1/\sqrt{p^2 + q^2}$ when $p, q \in Z$, $(p, q) = 1$, and $p^2 + q^2 > 1$.

If $N_i$ ($i = 1, 2$) are totally geodesic submanifolds of Riemannian manifolds $M_i$, then $N_1 \times N_2$ is a totally geodesic submanifold of $M_1 \times M_2$ and $\varepsilon(N_1 \times N_2) = \min(\varepsilon(N_1), \varepsilon(N_2)).$
2. **Theorem B.** Let $G$ be a group of isometries acting freely and properly discontinuously on a complete Riemannian manifold $M$. If $N$ is a complete $G$-invariant connected submanifold embedded in $M$ and $\bar{N} = \pi(N)$, where $\pi: M \to \bar{M} = M/G$ is the projection, then

$$\min\left\{ \frac{1}{2}d, \varepsilon(\bar{N}) \right\} \leq \varepsilon(N) \leq \varepsilon(\bar{N}),$$

where $d = \inf \{ d_M(x, gx); \ x \in N, \ g \in G, \text{ and } g \neq e \}$.

**Proof.** At first, we shall establish the equality

$$d_{\bar{M}}(\pi(x), \pi(y)) = \inf_{g \in G} d_M(x, yg), \quad x, y \in M.$$

Let us take points $x$ and $y$ of $M$ and put $\bar{x} = \pi(x)$, $\bar{y} = \pi(y)$. Since $M$ and $\bar{M}$ are complete and $\pi: M \to \bar{M}$ is a covering, there exist a curve $\bar{c}: \langle 0, 1 \rangle \to \bar{M}$ and its lift $c: \langle 0, 1 \rangle \to M$ such that $\bar{c}(0) = \bar{x}$, $\bar{c}(1) = \bar{y}$, $c(0) = x$, and $L(\bar{c}) = d_{\bar{M}}(\bar{x}, \bar{y})$. It is clear that $L(c) = L(\bar{c})$ and $c(1) = yg$ for some $g$ of $G$. Therefore,

$$d_{\bar{M}}(\bar{x}, \bar{y}) = L(\bar{c}) = L(c) \geq d_M(x, yg) \geq \inf_{g \in G} d_M(x, yg).$$

On the other hand, if $g \in G$, then there exists a curve $c: \langle 0, 1 \rangle \to M$ such that $c(0) = x$, $c(1) = yg$, and $L(c) = d_M(x, yg)$. Then

$$d_M(x, yg) = L(c) = L(\pi \circ c) \geq d_{\bar{M}}(\bar{x}, \bar{y}).$$

Inequalities (7) and (8) yield (6).

From (6), it follows immediately that

$$B_{\bar{M}}(\pi(x), r) = \pi(B_M(x, r))$$

for any $x$ of $M$ and $r > 0$.

From the assumptions of the Theorem, it follows that $G_N = \{ g \mid N; \ g \in G \}$ is a group of isometries of $N$ acting freely and properly discontinuously on $N$ in such manner that $\bar{N} = N/G_N$. Therefore, we can prove analogously to (6) and (9) that

$$d_{\bar{N}}(\pi(x), \pi(y)) = \inf_{g \in G} d_N(x, yg)$$

and

$$B_{\bar{N}}(\pi(x), r) = \pi(B_N(x, r))$$

for any $x \in N$, $y \in N$, and $r > 0$.

Comparing (9) and (11) we can conclude that if $x \in N$ and $r < \varepsilon(N)$, then $B_{\bar{N}}(\pi(x), r) = B_{\bar{M}}(\pi(x), r) \cap \bar{N}$. This yields the inequality $\varepsilon(N) \leq \varepsilon(\bar{N})$.

Using equalities (6) and (10), we can show that

$$\pi^{-1}(B_{\bar{M}}(\pi(x), r)) = \bigcup_{g \in G} B_M(xg, r)$$

6 — Annales Polonici Mathematici XLIV. 3.
and
\[ \pi^{-1} \left( B_N(\pi(x), r) \right) = \bigcup_{g \in G} B_N(xg, r) \]
for any \( x \) and \( r \). Therefore, if \( r < \varepsilon(\bar{N}) \) and \( x \in N \), then
\[ \bigcup_{g \in G} B_N(xg, r) = \pi^{-1} \left( B_{\bar{N}}(\pi(x), r) \right) \cap \pi^{-1}(\bar{N}) \]
\[ = \bigcup_{g \in G} (B_M(xg, r) \cap N). \]
If, in addition, \( r < \frac{1}{2}d \), then \( B_M(x, r) \cap B_M(xg, r) = \emptyset \) for any \( g \in G \), \( g \neq e \). This implies equality (1) for any \( x \) of \( N \) and \( r < \min \left( \frac{1}{2}d, \varepsilon(\bar{N}) \right) \). Consequently, we have the inequality \( \varepsilon(N) \geq \min \left( \frac{1}{2}d, \varepsilon(\bar{N}) \right) \) which completes the proof.

Let us note that if \( M \) is compact, then the number \( d \) in (5) is positive. Simple examples (geodesic lines on the cylinder and on the torus) show that inequalities (5) need not be satisfied if \( N \) is not \( G \)-invariant and that equalities \( \varepsilon(N) = \varepsilon(\bar{N}) \) and \( \varepsilon(N) = \frac{1}{2}d \) appear occasionally.

3. Let us recall that a submersion \( f: M \to B \), where \( M \) and \( B \) are Riemannian manifolds, is called Riemannian [7] if and only if \(|df(v)| = |v|\) for any vector \( v \) of \( TM \) orthogonal to \( \ker df \).

**Theorem C.** If \( f: M \to B \) is a Riemannian submersion with totally geodesic fibres and \( M \) is complete, then \( \varepsilon(N) = +\infty \) for any fibre \( N \) of \( f \).

**Proof.** For any smooth curve \( \gamma: \langle 0; 1 \rangle \to B \), \( \gamma(0) = x \), \( \gamma(1) = y \), let us define a mapping \( F_\gamma: f^{-1}(x) \to f^{-1}(y) \) as follows. If \( z \in f^{-1}(x) \), then there exists a curve \( \gamma_z: \langle 0; 1 \rangle \to M \) such that \( \gamma_z(0) = z \), \( f \circ \gamma_z = \gamma \) and \( \gamma'_z(t) \perp \ker df(\gamma_z(t)) \) for any \( t \) of \( \langle 0; 1 \rangle \). \( \gamma'_z \) is uniquely determined by these conditions and is called the horizontal lift of \( \gamma \). Put \( F_\gamma(z) = \gamma'_z(1) \).

The mappings \( F_\gamma \) are diffeomorphisms and, according to [3], a necessary and sufficient condition for \( F_\gamma \) to be isometries is that the fibres of \( f \) be totally geodesic.

Let us take an arbitrary curve \( c: \langle 0; 1 \rangle \to M \) and define a new curve \( C: \langle 0; 1 \rangle \to M \) putting
\[ C(t) = F_{\gamma_t}^{-1}(c(t)), \]
where \( \gamma_t: \langle 0; 1 \rangle \to B \) is given by \( \gamma_t(s) = f(c(st)) \). It is evident that \( C \) lies on the fibre \( f^{-1}(f(c(0))) \). The vector \( dF_{\gamma_t}(C(t)) \) is equal to the vertical component of \( \dot{c}(t) \). Therefore,
\[ |\dot{C}(t)| = |dF_{\gamma_t}(C(t))| \leq |\dot{c}(t)|, \quad t \in \langle 0; 1 \rangle, \]
and
\[ L(C) \leq L(c). \]
The above argumentation shows that for any curve on $M$ joining two points of a fibre $N$ of a submersion $f$ we are able to find a curve on $N$ which joins the same points and is shorter than the given one. Therefore, if $N$ is a fibre of $f$ and $x, y \in N$, then

$$d_N(x, y) \leq d_M(x, y).$$

Our Theorem follows immediately from this inequality.

4. Assume that $N$ is a submanifold of a Riemannian manifold $M$, the Ricci curvature $\operatorname{Ric}_M$ of $M$ is bounded by a positive number $k$ from below, and the diameter $d(N)$ of $N$ is greater than $\pi\sqrt{m-1}/\sqrt{k}$, where $m = \dim M$. Let us take points $x$ and $y$ of $N$ such that $d_N(x, y) > \pi\sqrt{m-1}/\sqrt{k}$. If $c: \langle 0; 1 \rangle \to M$ is a minimal geodesic on $M$ joining $x$ to $y$, then according to the well-known Myers theorem [6], $L(c) \leq \pi\sqrt{m-1}/\sqrt{k}$. Consequently, $a = d_N(x, y) - d_M(x, y) > 0$ and

$$y \in B_M(x, d_N(x, y) - b) \cap N - B_N(x, d_N(x, y) - b)$$

for any $b$ of $(0; a)$. In this manner, we established the following:

**Proposition D.** If $N$ is a submanifold of a complete $m$-dimensional Riemannian manifold $M$ and the Ricci curvature of $M$ is bounded by a number $k > 0$ from below, then either $d(N) \leq \pi\sqrt{m-1}/\sqrt{k}$ or $\varepsilon(N) \leq \pi\sqrt{m-1}/\sqrt{k}$.

Replacing in the above argumentation the classical Myers theorem by its generalization due to Galloway [1] we can generalize Proposition D as follows:

**Proposition D'.** Assume that $M$ is a complete $m$-dimensional Riemannian manifold and that there exist constants $k > 0$ and $c \geq 0$, and a differentiable function $h: M \to \mathbb{R}$ such that $|h| \leq c$ and

$$\operatorname{Ric}_M(v, v) \geq k + v(h)$$

for any unit vector $v$ of $TM$. Then the inequality

$$\min(d(N), \varepsilon(N)) \leq \frac{\pi}{k}(c + \sqrt{c^2 + k(m-1)})$$

holds for any submanifold $N$ of $M$.

Proposition D' and Theorem C imply the following:

**Corollary.** Under the hypotheses of Proposition D', any fibre $N$ of a Riemannian submersion $f: M \to B$ with totally geodesic fibres satisfies the inequality

$$d(N) \leq \frac{\pi}{k}(c + \sqrt{c^2 + k(m-1)}).$$
**Examples.** In the case of the Riemannian submersion \( f: P^{2n+1} \rightarrow P^n Q \)
eq 0 and \( k = \frac{1}{2} (n+1) \) is not informative: The fibres of \( f \) are isometric to \( S^2 \) and have diameter equal to \( \pi \) while the right-hand side of (12) equals \( \pi \sqrt{4n-2}/(n+1) \) and tends to \( 2\pi \) as \( n \to \infty \). The situation is different in the case of the orthogonal group \( O(n) \) equipped with the standard biinvariant Riemannian metric. If \( n < m \), then \( O(n) \) is a closed subgroup of \( O(m) \) and the projection \( O(m) \to O(m)/O(n) \) is a Riemannian submersion with totally geodesic fibres isometric to \( O(n) \). The Ricci curvature of \( O(m) \) is constant and equals \( \frac{1}{2} (m-1)(m-2) \). From (12), it follows that

\[
d(O(n)) \leq \pi \sqrt{m/(m-2)}
\]

for any \( m > n \). Passing with \( m \) to the infinity, we get

\[
d(O(n)) \leq \pi.
\]

On the other hand, \( d(O(n)) \geq d(O(2)) = \pi \). It follows that \( d(O(n)) = \pi \) for any \( n \geq 2 \).

5. Let \( F \) be a foliation of a Riemannian manifold \( M \). If all the leaves of \( F \) are compact minimal submanifolds of \( M \), then \( F \) is stable, i.e. the quotient \( M/F \) is Hausdorff [8]. If \( X \) is an arbitrary subset of \( M \) saturated by compact minimal leaves, then the quotient \( X/F \) need not be Hausdorff even if all the leaves of \( F \) are totally geodesic. For example, if \( M = R \times S^1 \times S^1 \) (endowed with the standard Riemannian metric), \( F \) is the 1-dimensional foliation of \( M \) defined by the vector field

\[
Z = t \frac{\partial}{\partial x} + \frac{\partial}{\partial y},
\]

where \((t, x, y)\) are standard coordinates on \( M \), and \( X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots \} \times \times S^1 \times S^1 \), then \( X \) is saturated by closed geodesics but \( X/F \) is not Hausdorff. In [9], we proved the following:

**Proposition E.** If \( X \subset M \) is a set saturated by compact totally geodesic leaves of a foliation \( F \) of a Riemannian manifold \( M \) and the function

\[
X \ni x \mapsto \varepsilon(L_x),
\]

where \( L_x \) denotes the leaf of \( F \) passing through \( x \), is locally bounded by positive numbers from below (i.e., for any \( x \) of \( X \) there exist a neighbourhood \( U \) of \( x \) and a number \( a > 0 \) such that \( \varepsilon(L_y) \geq a \) for any \( y \) of \( U \cap X \), then \( X/F \) is Hausdorff.

The converse is not true. For example, if \( F \) is the standard foliation of the Möbius strip \( M \) by closed geodesics (Figure 1), then \( M/F \) is Hausdorff but the function (13) is not bounded from below by any positive number in any neighbourhood of the "central leaf" \( L_0 \). In fact, if \( L \in F \) and \( d(L, L_0) = a > 0 \) is sufficiently small, then \( \varepsilon(L) = 2a \).
6. A manifold $M$ with a projective structure can be equipped with a projective invariant pseudometric $\delta_M$ (see [4], [5], [10]). If $\delta_M$ is a metric, then $M$ is said to be hyperbolic. Let $N$ be a submanifold of a hyperbolic manifold $M$. If $N$ carries the projective structure induced from $M$, then $N$ is hyperbolic and

$$\delta_N(x, y) \geq \delta_M(x, y)$$

for all $x$ and $y$ of $N$. It follows that the balls $B_N(x, r) = \{y \in N; \delta_N(x, y) < r\}$ and $B_M(x, r) = \{y \in M; \delta_M(x, y) < r\}$ satisfy condition (2) for any $x$ of $N$ and $r > 0$. One can expect that equality (1) holds for sufficiently small $r$ in this case. The following example shows that this is not true even if $M$ and $N$ are domains on the plane.

**Example.** Let us consider the situation described in Figure 2. $M$ is a convex bounded domain (a disc of the radius 2) on the plane. Therefore, $M$ is hyperbolic and the metric $\delta_M$ is given by

$$\delta_M(z_1, z_2) = \left| \log \frac{(z_1 - t_2)(z_2 - t_1)}{(z_1 - t_1)(z_2 - t_2)} \right|,$$
where \( t_1 \) and \( t_2 \) are the points of intersection of the boundary of \( M \) with the line passing through \( z_1 \) and \( z_2 \). Exactly the same can be said about the domain \( N \) (a disc of the radius 1). Therefore,

\[
\delta_M(x, y) = \log \frac{(3-a)(1+a+b)}{(1+a)(3-a-b)}
\]

and

\[
\delta_N(x, y) = \log \frac{1+b}{1-b}.
\]

It is easy to see that for any \( r > 0 \) there exists a point \( y \) of \( l \cap N \) such that

\[
\delta_N(x, y) > r > \delta_M(x, y).
\]

References


POLISH ACADEMY OF SCIENCES
MATHEMATICAL INSTITUTE

Reçu par la Rédaction le 19. 10. 1981