Bergman completeness of complete circular domains

by M. Jarnicki (Kraków) and P. Pflug (Vechta)

Abstract. It is proved that any bounded complete circular domain of holomorphy with continuous Minkowski functional is complete w.r.t. the Bergman distance.

1. Introduction. We start with the basic notions we shall use:

Definition. A domain $D \subset \mathbb{C}^n$ is called complete circular if whenever $z \in D, \lambda \in \mathbb{C}, |\lambda| \leq 1$ then $\lambda z \in D$.

We mention that any complete circular domain can be defined by its Minkowski functional, i.e., there exists $h: \mathbb{C}^n \to [0, \infty)$. $h(\lambda z) = |\lambda| h(z)$ ($\lambda \in \mathbb{C}, z \in \mathbb{C}^n$) such that $D = \{z \in \mathbb{C}^n: h(z) < 1\}$; sometimes we write $D = D_h$. By [2], we know: $D_h$ is a domain of holomorphy iff $h$ is a plurisubharmonic function.

Observe that any pseudoconvex Reinhardt domain $D$ containing the origin is a complete circular domain $D = D_h$ with continuous Minkowski functional $h$.

Let $D$ be a domain in $\mathbb{C}^n$; then by $c_D(z', z'')$ we denote the Carathéodory pseudodistance between $z', z''$ (cf. [3]). Then the following is known [7]:

Theorem. Any bounded pseudoconvex Reinhardt domain $D \subset \mathbb{C}^n$, containing the origin, is $c_D$-finitely compact, i.e., any $c_D$-ball is a relatively compact subset of $D$ w.r.t. the $\mathbb{C}^n$-topology.

Remark. There is also the notion of $c_D$-completeness in the sense of metric spaces. But, so far, it is not known whether both notions coincide. It is clear that $c_D$-finitely compact always implies $c_D$-complete.

Hence, any such bounded pseudoconvex Reinhardt domain is the existence domain of a bounded holomorphic function. On the other hand, one can characterize the bounded complete circular domains of holomorphy which are $H^\infty$-domains of holomorphy (cf. [8]).

Theorem. Let $D = D_h$ be a pseudoconvex, bounded complete circular domain in $\mathbb{C}^n$. Then the following conditions are equivalent:
(a) $D$ is an $H^\infty$-domain of holomorphy,
(b) \( |z \in \mathbb{C}^n; h \text{ is not continuous at } z \) is pluripolar.

Hence, not every bounded complete circular domain $D = D_h$ of holomorphy is $c_B$-finitely compact. Moreover, it is clear (cf. [2]) that continuity of $h$ is a necessary condition: this leads to the following question: is any bounded, complete circular domain $D = D_h$ of holomorphy with continuous Minkowski functional already $c_B$-finitely compact?

In view of Hahn's inequality [4], necessary for having a positive answer is that every such $D = D_h$ is Bergman complete. Here we give a proof of this result, which was found in February 1986 while the second author was a guest at Jagiellonian University, Kraków.

2. Bergman completeness. Let $D$ be a bounded domain in $\mathbb{C}^n$ with its Bergman kernel $K_D: D \times D \to \mathbb{C}$, holomorphic in the first, antiholomorphic in the second variable. By

$$B_D(z, X) := \left( \sum \frac{\partial^2}{\partial z_\mu \overline{z}_\nu} \log K_D(z, z) X_\mu \overline{X}_\nu \right)^{1/2} \quad (z \in D, x \in \mathbb{C}^n),$$

we denote the Bergman metric in $D$ and by

$$h_D(z', z'') := \inf_{0 \leq t_1 \leq \cdots \leq t_k} \left\{ \int_0^1 \| B_D(\gamma(t), \gamma'(t)) dt \mid \gamma \text{ piecewise } C^1 \text{-curve in } D \text{ from } z' \text{ to } z'' \right\} \quad (z', z'' \in D)$$

the Bergman distance in $D$.

We also recall the Skwarczyński distance for $z', z'' \in D$ [9]:

$$\hat{d}_D(z', z'') := \left[ 1 - \frac{|K_D(z', z'')|}{\sqrt{K_D(z', z') \cdot K_D(z'', z'')}} \right]^{1/2}$$

and observe: $\hat{d}_D \leq c h_D$ (cf. [5]), where $c$ is a constant.

Thus, in order to prove that $D$ is $h_D$-complete is suffices to show that $D$ is $\hat{d}_D$-complete.

More precisely, we use the following result [9]:

**Theorem.** Let $D$ be a bounded domain in $\mathbb{C}^n$ and assume:
(i) the bounded holomorphic functions in $D$ lie dense in the Hilbert space $L^2(D)$ of square-integrable holomorphic functions,
(ii) $\lim_{\zeta \to \xi} K_D(z, z) = \infty$ whenever $z \in \partial D$.

Then $D$ is $\hat{d}_D$-complete.

Observe that, for a bounded complete circular domain $D = D_h$ with continuous $h$, condition (i) is always fulfilled: take $f_t(z) := f(t \cdot z)$ for $t \to 1$. 
What remains is the verification of condition (ii). Here we use the following deep result due to Ohsawa-Takegoshi [6]:

**Theorem.** Let \( \Omega \) be a bounded pseudoconvex domain in \( \mathbb{C}^n \), \( \psi : \Omega \to \mathbb{R} \cup \{ -\infty \} \) a plurisubharmonic function and \( H \subset \mathbb{C}^n \) a complex hyperplane. Then, for any holomorphic function \( f \) on \( \Omega \cap H \) satisfying \( \int_{\Omega \cap H} e^{-\psi} |f|^2 \, dV_{n-1} < \infty \), there exists a holomorphic function \( F \) on \( \Omega \) with \( F|_{\Omega \cap H} = f \) and \( \int_{\Omega \cap H} e^{-\psi} |F|^2 \, dV_n \leq C \int_{\Omega \cap H} e^{-\psi} |f|^2 \, dV_{n-1} \), where \( C = C(\text{diam} \, \Omega) = 1620 \pi [1 + (\text{diam} \, \Omega)^2] \) and \( dV_k \) denotes the \( 2k \)-dimensional Lebesgue measure.

Now, we can formulate and prove our result:

**Theorem.** Let \( D \) be a bounded complete circular domain of holomorphy with its Minkowski functional \( h \) and assume \( h \) to be continuous. Then \( D \) is \( \lambda \)-complete and hence Bergman complete.

**Proof.** We only have to verify that \( \lim K_D(z, z) = +\infty \) whenever \( z \in \partial D \).

Therefore we claim: for any \( D = D_h \subset \mathbb{C}^n \) there exists \( h > 0 \), depending only on the diameter and the dimension, such that, for \( z \in D \).

\[ K_D(z, z) \geq h/[1 - h^2(z)]^2. \]

In order to prove (*) we proceed by induction:

In the case \( n = 1 \), we always have \( h(z) = \frac{1}{R} |z| \) and if \( z \in D \).

\[ K_D(z, z) = \frac{1}{\pi R^2} \frac{1}{[1 - h^2(z)]^2}. \]

Now we turn to the step \((n-1)\) to \( n \). Thus we assume \( D = D_h \subset \mathbb{C}^n \). Let \( z^0 \in D = D_h, z^0 \neq 0 \): without loss of generality we may assume \( z^0 = (z^0_1, 0, \ldots, 0) \). Define \( H := \{ z \in \mathbb{C}^n : z_n = 0 \} \). Then there exists \( C_0 > 0 \) only depending on the diameter of \( D \) such that every holomorphic function on the bounded complete circular pseudoconvex domain \( D \cap H \) in \( \mathbb{C}^{n-1} \), square-integrable w.r.t. the \((2n-2)\)-dimensional Lebesgue measure, extends to \( F \) holomorphic on \( D \) and satisfying \( \| F \|_{L^2(D)} \leq C_0 \| f \|_{L^2(D \cap H)} \).

Therefore we obtain

\[ K_D(z^0, z^0) \geq \frac{|F(z^0)|^2}{\| F \|_{L^2(D)}^2} \geq \frac{|f(z^0)|^2}{C_0^2 \| f \|_{L^2(D \cap H)}^2}. \]

Since \( f \in L^2(D \cap H) \) was arbitrary, this inequality implies

\[ K_D(z^0, z^0) \geq \frac{1}{C_0^2} K_{D \cap H}(z^0, z^0) \quad (z^0 = (z^0_1, 0)). \]
and, by the induction hypothesis,

\[ K_D(z^0, z^0) \geq h/[1 - h^2(z^0)]^2. \]

Hence \((*)\) is verified.

Applying that \(h\) is a continuous function, we immediately conclude
\[ \lim_{z \to z^0} K_D(z, z) = \infty. \] whenever \(\zeta \in \partial D\) which proves our theorem.

**Remarks.** We like to mention that this result is also formulated in [1], Theorem 1, 11, but the arguments given there are incomplete.

We close this note by posing the following question:

Let \(\tilde{h}: C^n \to [0, \infty)\) be a plurisubharmonic function with \(\tilde{h}(iz) = |\lambda| \tilde{h}(z)\) \((z \in C^n, \lambda \in \mathbb{C})\) such that \(\tilde{h}\) vanishes on a dense subset of \(C^n\) (cf. [8]). We define \(D = D_h := \{z \in C^n : h(z) < 1\}\), where \(h(z) := \tilde{h}(z) + |z|^2\). Hence \(D\) is a pseudoconvex, complete circular domain with \(\bar{D} = B\), \(B\) the open Euclidean unit ball in \(C^n\), and its Minkowski functional is far away from being continuous. Therefore we ask:

Is \(D\) also Bergman complete?

**References**


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