Restricted homogeneity implies bi-additivity

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Abstract. Let $R$ be a commutative ring with identity, and let $M$ be an $R$-module. Suppose $F: M \times M \to R$ satisfies

\begin{equation}
F(x, y) + F(x + y, z) = F(x, y + z) + F(y, z)
\end{equation}

and

\begin{equation}
F(0, 0) = 0.
\end{equation}

Two results are proved.

**Theorem 1.** Let $r, s$ be fixed elements of $R$ such that $rs$ is not a zero divisor, and $r - s$ is an invertible element of $R$. If $F$ satisfies (1), (2) and

\begin{equation}
F(rx, sy) = rsF(x, y),
\end{equation}

then $F$ is additive in each variable.

**Theorem 2.** Let $R$ be the ring $\mathbb{Z}$ of rational integers. Let $r, s$ be distinct non-zero integers. If $F$ satisfies (1), (2) and (3), then $F$ is bi-additive.

Theorem 2 yields the result due to Jordan and von Neumann, that if $f: M \to \mathbb{Z}$ satisfies the parallelogram law, then $F(x, y) = f(x + y) - f(x) - f(y)$ is bi-additive.

A commonly accepted definition of quadratic forms is the following (see e.g. Jacobson [1], Definition 6.1), where $R$ is a commutative ring with identity, and $M$ is a (unitary) $R$-module. A function $f: M \to R$ is a quadratic form if

\begin{enumerate}
    \item $f(rx) = r^2 f(x)$ for all $r \in R, x \in M,$
    \item $F: M \times M \to R$ is bilinear, where, for all $x, y \in M,$
\end{enumerate}

\begin{equation}
F(x, y) := f(x + y) - f(x) - f(y).
\end{equation}

The second requirement can be separated into two parts: that $F$ be homogeneous (of degree one) in each variable, and that $F$ be bi-additive. It will be shown that suitable homogeneity of $F$, and the fact (a consequence of its definition by (1)) that $F$ satisfies

\begin{equation}
F(x, y) + F(x + y, z) = F(x, y + z) + F(y, z)
\end{equation}

implies that $F$ is bi-additive.
The particular situation which motivates and illuminates our treatment is the deduction of bi-additivity from the parallelogram law (cf. Jordan-von Neumann [2], Theorem 1). Suppose \( f : M \to R \) satisfies
\[
 f(x+y) + f(x-y) = 2f(x) + 2f(y)
\]
and 2 is not a zero divisor in \( R \), then \( F \) satisfied the partial homogeneity condition
\[
 F(x, -y) = -F(x, y).
\]

Bi-additivity of \( F \) is proved in

**Proposition 1.** Let \( R \) be a ring in which 2 is not a zero divisor. Let \( M \) be an \( R \)-module and suppose \( F : M \times M \to R \) satisfies (2) and (4). Then \( F \) is bi-additive.

**Proof.** Substitute \(-z\) for \( z \) in (2) to obtain
\[
 F(x, y) + F(x+y, -z) = F(x, y-z) + F(y, -z).
\]
Add (2) and (5), and use (4) to deduce that
\[
 2F(x, y) = F(x, y+z) + F(x, y-z).
\]
In (6) interchange \( y \) and \( z \), and add the resulting equation to (6) to deduce (using (4) again) that
\[
 2F(x, y) + 2F(x, z) = 2F(x, y+z).
\]
Since 2 is not a zero divisor, (7) yields the additivity of \( F \) in the second variable. A further use of (2) yields the additivity of \( F \) in the first variable. Hence \( F \) is bi-additive.

This result is generalized in

**Proposition 2.** Let \( r, s \) be fixed elements of \( R \) such that \( rs \) is not a zero divisor. If \( F : M \times M \to R \) satisfies (2)
\[
 F(rx, sy) = rsF(x, y)
\]
and
\[
 F(0, 0) = 0
\]
then
\[
 F(x+ty, z) = F(x, z) + F(ty, z)
\]
for all \( x, y, z \in M \), where \( t := r-s \).

**Remark.** If \( rs-1 \) is not a zero divisor, then (9) is a consequence of (8). If \( F \) satisfies (2) and (8), then \( F-F(0, 0) \) satisfies (2), (8) and (9).
Proof. In (2) replace $x$ by $rx$, $y$ by $rsy$, and $z$ by $sz$ to obtain
\[(11) \quad F(rx, rsy) + F(rx + rsy, sz) = F(rx, rsy + sz) + F(rsy, sz).\]
Simplify (11) using (8) and cancel the $rs$ factors resulting to deduce that
\[(12) \quad F(x, ry) + F(x + sy, z) = F(x, ry + z) + F(sy, z).\]
However, writing $ry$ for $y$ in (2) yields
\[(13) \quad F(x, ry) + F(x + ry, z) = F(x, ry + z) + F(ry, z).\]
Subtracting (12) from (13), and using the notation $t := r - s$ we have
\[(14) \quad F(x + sy + ty, z) - F(x + sy, z) = F(sy + ty, z) - F(sy, z).\]
Set $x = -sy$ in (14), and use the fact that $F(x', 0) = F(0, z') = 0$ by (9) and (2), to obtain
\[F(sy + ty, z) - F(sy, z) = F(ty, z); \text{ so (14) can be rewritten}\]
\[(15) \quad F(x + sy + ty, z) - F(x + sy, z) = F(ty, z).\]
Finally in (15) replace $x + sy$ by $x$ to deduce (10).

Corollary. Suppose $F$ satisfies (2), (8), and (9); and $r - s$ is an invertible element of $R$, then $F$ is bi-additive. In particular, if $R$ is a field with at least 3 elements and $F$ satisfies (2), (8) and (9) for a pair $r \neq 0$, $s \neq 0$, $r \neq s$, then $F$ is bi-additive.

Proof. Write $t^{-1}y$ for $y$ in (10).

If one considers the corollary applied to $r = 1$, $s = -1$ one sees that one has to assume that 2 is invertible to deduce the additivity of $F$, whereas in Proposition 1 all one requires in that 2 not be a zero divisor. It seems that the fact that $r$ and $s$ are in the subring of $R$ generated by 1 is critical, as is shown in the next (and final) result.

Proposition 3. Let $r, s$ be fixed elements of $Z$ (the ring of rational integers) such that $rs \neq 0$, and $r \neq s$. If $F: M \times M \to Z$ satisfies (2), (8) and (9), then $F$ is bi-additive.

Proof. We can assume that $|rs| > 1$, as $rs = 1$ violates the assumption $r \neq s$, and $rs = -1$ is taken care of in Proposition 1.

We exploit (10) by writing ($t := r - s$ as usual). 
\[(16) \quad rsF(x, y) = F(rx, sy) = F(sx + tx, sy) = F(sx, sy) + F(tx, sy).\]
Now by (2)
\[F(tx, y) + F(tx + y, z) = F(tx, y + z) + F(y, z)\]
so using (10) here again
\[(17) \quad F(tx, y) + F(tx, z) = F(tx, y + z).\]
Hence by (17), and the fact that \( s \in Z \)

\[
F(tx, sy) = sF(tx, y).
\]

Indeed, for all \( m, n \in Z \),

\[
F(mtx, nty) = mnF(tx, y).
\]

The outcome of all this is that (16) may be rewritten as

\[
F(sx, sy) = rsF(x, y) - sF(tx, y).
\]

For each \( n \geq 0 \) define \( \alpha_n, \beta_n \in Z \) by \( \alpha_0 = 1, \beta_0 = 0 \) and

\[
\alpha_{n+1} = rs \cdot \alpha_n, \quad \beta_{n+1} = rs \cdot \beta_n + s^{2n+1}
\]

for \( n \geq 1 \). Then it is an easy exercise to use (20) and (21) to prove by induction, that for all \( n \geq 0 \)

\[
F(s^n x, s^n y) = \alpha_n F(x, y) - \beta_n F(tx, y).
\]

It is also easy to prove from (21) that

\[
\alpha_n - t\beta_n = s^n n.
\]

Since \( t \neq 0 \), there are by Euler's theorem (or the fact that \( Z/tZ \) is a finite ring), positive integers \( m > n \) such that \( s^m \equiv s^n \mod t \) say \( s^m = s^n + ut \). Then on the one hand by (22)

\[
F(s^m x, s^m y) = \alpha_m F(x, y) - \beta_m F(tx, y)
\]

and on the other hand, using (10),

\[
F(s^m x, s^m y) = F(s^n x + ut, s^n y + uty)
\]

\[
= F(s^n x, s^n y) + F(s^n x, uty) + F(utx, s^n y) + F(utx, uty)
\]

\[
= \alpha_n F(x, y) - \beta_n F(tx, y) + us^n F(x, ty) +
\]

\[
+ us^n F(tx, y) + u^2 tF(tx, y)
\]

by repeated use of (17). Moreover, \( F(tx, ty) = tF(tx, y) \) and \( F(tx, ty) = tF(x, ty) \), by (19). Since \( t \neq 0 \), we see that \( F(tx, y) = F(x, ty) \). Thus our second evaluation of \( F(s^m x, s^m y) \) is \( \alpha_n F(x, y) - (\beta_n - 2us^n - u^2 t) = F(tx, y) \).

Equating the two evaluations, and rearranging, we obtain

\[
(\beta_m - \beta_n + 2us^n + u^2 t)F(tx, y) = (\alpha_m - \alpha_n) F(x, y).
\]

Multiply both sides of this by \( t \), and use (23) to deduce.

\[
(\alpha_m - \alpha_n) F(tx, y) = (\alpha_m - \alpha_n) tF(x, y).
\]
Finally, $x_m - x_n = (rs)^m - (rs)^n \neq 0$ since $|rs| > 1$ and $m > n$. So we deduce from (25) that

(26) \[ F(tx, y) = tF(x, y). \]

Now use (17) and (26) to infer that

\[ tF(x, y+z) = tF(x, y) + tF(x, z); \]

cancelling $t$ (since $t$ is not a zero divisor) yields the desired result.

References


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