RECTILINEARLY AND RECTIFIABLY AMBIGUOUS POINTS OF A FUNCTION HARMONIC INSIDE A SPHERE

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Denote the Cartesian coordinates of a point in three-dimensional Euclidean space by \( x, y, z \), and set

\[
S = \{(x, y, z): x^2 + y^2 + z^2 < 1\},
\]

\[
T = \{(x, y, z): x^2 + y^2 + z^2 = 1\}.
\]

Suppose that \( f \) is a single-valued, real-valued function defined for every point \( P \in S \). A point \( Q \in T \) is called an ambiguous point of \( f \), if there exist Jordan arcs \( J_1 \) and \( J_2 \) that lie in \( S \) except for their common end point \( Q \), on which the limits

\[
\lim_{P \to Q} f(P) \quad \text{and} \quad \lim_{P \to Q} f(P)
\]

exist and are unequal. Such arcs \( J_1 \) and \( J_2 \) are called arcs of ambiguity of \( f \) at \( Q \). If \( f \) has a pair of rectilinear arcs of ambiguity at \( Q \), then \( Q \) is called a rectilinearly ambiguous point of \( f \).

**Theorem.** There exists a harmonic function \( h(P) \) \( (P \in S) \) and an everywhere dense subset \( D \) of \( T \) with \(|D| = 2^{\aleph_0}\) such that every point of \( D \) is a rectilinearly ambiguous point of \( h \) and every point \( Q \in T \setminus D \) is an ambiguous point of \( h \) with arcs of ambiguity \( J_1^Q \) and \( J_2^Q \) at \( Q \) such that \( J_1^Q \) is a rectilinear segment and \( J_2^Q \) is a rectifiable Jordan arc.

**Proof.** To simplify the description of our construction, it is convenient to work initially with the cube

\[
A = \{(x, y, z): 0 < x < 1, 0 < y < 1, 0 < z < 1\}
\]
instead of $S$, and confine our attention to the face

$$F = \{(x, y, 0) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$$

instead of $T$. In the final stages of the proof, we shall suppose that the construction has been carried out for $S$ and $T$, which entails no conceptual difficulty.

We form two sets, $G_1$ and $G_2$, in $A$.

To define $G_1$, consider the square (interior and boundary) $W$ with vertices $(\frac{1}{8}, \frac{1}{8}, \frac{1}{2}), (\frac{3}{8}, \frac{1}{8}, \frac{1}{2}), (\frac{3}{8}, \frac{7}{8}, \frac{1}{2})$ and $\left(\frac{1}{8}, \frac{7}{8}, \frac{1}{2}\right)$. Construct (see, e.g., [4, p. 135]) the familiar perfect nowhere dense subset $V$ of $W$ by first dividing $W$ into nine equal squares, retaining the four at the corners of $W$, and labeling these four $W_1, W_2, W_3, W_4$ in the same order as the corresponding vertices of $W$ were listed above. Then divide each $W_j (j = 1, 2, 3, 4)$ into nine equal squares, retain the four at the corners of $W_j$, and label these four $W_{j1}, W_{j2}, W_{j3}, W_{j4}$, in the same order as before. Continuing in this way, we obtain for every sequence $k_1, \ldots, k_n$, where each $k_m (m = 1, 2, \ldots, n)$ is one of the numbers $1, 2, 3, 4$, a square $W_{k_1k_2\ldots k_n}$. Then the set $V$ is the set of all points of the form

(1) 

$$W_{k_1} \cap W_{k_1k_2} \cap \ldots \cap W_{k_1k_2\ldots k_n} \cap \ldots$$

where $k_1, k_2, \ldots, k_n, \ldots$ is any infinite sequence whose terms belong to the set \{1, 2, 3, 4\}. Next we divide the square $F$ into four equal squares $F_1, F_2, F_3, F_4$, then divide each $F_j (j = 1, 2, 3, 4)$ into four equal squares $F_{j1}, F_{j2}, F_{j3}, F_{j4}$, and so on, each time labeling them in the same order as the corresponding subsquares of $W$. Join the point (1) of the set $V$ to the point

(2) 

$$F_{k_1} \cap F_{k_1k_2} \cap \ldots \cap F_{k_1k_2\ldots k_n} \cap \ldots$$

of the square $F$ by a rectilinear segment. This segment shall contain the point (1) but not the point (2). Define $G_1$ to be the union of all such segments.

The construction of $G_2$ takes place in the truncated pyramid $M$ whose base is $F$ and whose upper base is $W$. The interior and boundary of the pyramid, with the exception of the base $F$, are regarded as belonging to $M$. Let

$$\frac{7}{8} > s_2 > s_3 > \ldots > s_n > s_{n+1} > \ldots > 0, \quad \lim_{n \to \infty} s_n = 0.$$ 

For each $n = 1, 2, 3, \ldots$ we define $2^n$ planes. For $n = 1$, each of the two planes is determined by the center of the square $W$ and one of the two lines dividing the square $F$ into the four equal squares $F_j (j = 1, 2, 3, 4)$, where we consider only that part of the plane that belongs to $M$. For $n > 1$, each of the $2^n$ planes is determined by the center of a square $W_{k_1k_2\ldots k_n}$ and one of the two lines dividing the square $F_{k_1k_2\ldots k_n}$ into the four equal squares.
$F_{k_1...k_{n-1}j} (j = 1, 2, 3, 4)$, where we consider only that part of the plane lying in the truncated pyramid

$$M \cap \{(x, y, z): 0 < z \leq s_n\}.$$

Define $G_2$ to be the union of all such parts of planes.

It is evident from the construction of $G_1$ and $G_2$ that $G_1 \cap G_2 = \emptyset$. Let $D$ be the union of the segments used above to divide $F$, each $F_{k_1}$, each $F_{k_1k_2}$, ..., each $F_{k_1...k_{n-1}j}$, ..., into four equal squares. Then clearly $|D| = 2^{n_0}$ and $D$ is an everywhere dense subset of $F$. If $Q \in D$, then there is a rectilinear segment at $Q$ belonging to $G_1$; and there is a plane, and hence a rectilinear segment, at $Q$ belonging to $G_2$. If $Q \in F \setminus D$, then there is a rectilinear segment at $Q$ belonging to $G_1$; and there is a simple polygonal rectifiable arc at $Q$ that is a subset of $G_2$.

Turning now to $S$ and $T$, we may assume that we have constructed two sets, $G_1$ and $G_2$, in $S$, where $G_1$ is the union of rectilinear segments and $G_2$ is the union of parts of planes, and an everywhere dense subset $D$ of $T$, for which the assertions in the preceding paragraph hold.

Let $f(P)$ ($P \in S$) be a real-valued continuous function mapping $S$ onto the unit interval in such a way that $f(G_1) = 0$ and $f(G_2) = 1$. Set $G = G_1 \cup G_2$.

Let

$$0 < r_0 < r_1 < \ldots < r_n < r_{n+1} < \ldots < 1, \quad \lim_{n \to \infty} r_n = 1,$$

and, for $n = 0, 1, 2, \ldots$, set

$$S_n = \{(x, y, z): x^2 + y^2 + z^2 < r_n^2\},$$

$$T_n = \{(x, y, z): x^2 + y^2 + z^2 = r_n^2\},$$

$$K_n = (S_n \cup T_n \cup G) \cap (S_{n+1} \cup T_{n+1}).$$

The next step is potential-theoretical. It is clear that $K_n$ is a compact set; denote its complement by $CK_n$. Consider any boundary point of $K_n$ (such a boundary point is also a boundary point of $CK_n$). A sphere with this point as center and radius $\varrho$ contains on its surface a continuum belonging to $CK_n$ with diameter greater than $\varrho$, as is evident from the construction of $G$. It follows [5, p. 294, Theorem 5.4] that every boundary point of $K_n$ is a regular point of $CK_n$. Consequently ([5, p. 308, Theorem 5.10]) $CK_n$ is not thin at any boundary point, and so ([2, p. 60]) $K_n$ has no unstable boundary point. Therefore ([3]) any continuous function on $K_n$ that is harmonic at every interior point of $K_n$ can be uniformly approximated on $K_n$ as closely as desired by a harmonic polynomial.

With this in hand, it is now possible to construct, by a method like that
employed in [1, pp. 153–154], a harmonic function $h(P) (P \in S)$ such that

$$\lim_{P \to T \atop P \in G} [h(P) - f(P)] = 0,$$

which implies our theorem. □

Remark. It would be interesting to know if there exists a harmonic function $h(P) (P \in S)$ such that every point of $T$ is a rectilinearly ambiguous point of $h$.

REFERENCES


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