ON UNIVERSAL ALGEBRAS
HAVING BASES OF DIFFERENT CARDINALITIES

BY

J. DUDEK (WROCLAW)

The aim of this note is to prove some remarks for abstract algebras with two bases of different cardinalities. Our terminology and notation are standard (see [2] and [3]).

In [3] Marczewski has proposed the following conjecture:

If an algebra \( \mathfrak{A} \) has two bases of different cardinalities, then for each natural number \( n \) there exists an essentially \( n \)-ary algebraic operation over \( \mathfrak{A} \). (For partial results see [1] and [4].)

The results of this paper are not quite related to Marczewski’s conjecture, they are concerned with algebras of a variety \( \mathcal{A}_{m,n} \) defined in the sequel.

Let \( \{ a_1, a_2, \ldots, a_n \} \) and \( \{ b_1, b_2, \ldots, b_m \} \) be bases of an algebra \( \mathfrak{A} = (A, F) \). We assume that \( m < n \). It is easy to see that there exist unique algebraic operations \( f_i \) and \( g_j \) over \( \mathfrak{A} \) such that

\[
    b_j = f_j(a_1, a_2, \ldots, a_n) \quad \text{and} \quad a_i = g_i(b_1, b_2, \ldots, b_m)
\]

for \( j = 1, 2, \ldots, m \) and \( i = 1, 2, \ldots, n \).

Then the algebra \( \mathfrak{A}_{m,n} = (A, f_1, f_2, \ldots, f_m, g_1, g_2, \ldots, g_n) \) belongs to the variety \( \mathcal{A}_{m,n} \) defined by the following identities:

\[
    (\ast) f_j(g_1, g_2, \ldots, g_n) = \phi_j^m, \\
    (\ast\ast) g_i(f_1, f_2, \ldots, f_m) = \phi_i^n.
\]

The following notation will be used in the sequel: for any algebra \( \mathfrak{A} \) let \( \mathcal{S}(\mathfrak{A}) \) denote the set of all \( n \)'s such that in \( \mathfrak{A} \) there exists an essentially \( n \)-ary algebraic operation.

If \( F^* \subseteq A(F) \), then the algebra \( (A, F^*) \) is said to be a reduct of a given algebra \( \mathfrak{A} = (A, F) \). The most interesting reducts of an algebra \( \mathfrak{A} \) are: the trivial reduct \( (A, \emptyset) \), the idempotent reduct \( I(\mathfrak{A}) = (A, I(F)) \), where \( I(F) \) consists of all idempotent algebraic operations (i.e., \( f \in I(F) \) if, for \( x \in A \), \( f(x, x, \ldots, x) = x \)), and \( S(\mathfrak{A}) = (A, S(F)) \), where \( S(F) \) denotes the set of all completely symmetric algebraic operations over \( \mathfrak{A} \). The algebras \( I(\mathfrak{A}) \) and \( S(\mathfrak{A}) \) were investigated by Urbanik in [6] and [7].
PROPOSITION 1. For any non-one-element algebra \( A \in \mathcal{A}_{1,n} \) and any positive integer \( s \) there exists an essentially \( s \)-ary algebraic operation \( d(x_1, x_2, \ldots, x_n) \) over \( A \) such that \((A, d)\) is an \( s \)-dimensional diagonal algebra.

COROLLARY 1. If \( A \) has bases consisting of 1 and \( m \geq 2 \) elements, then for all \( s \) it contains an essentially \( s \)-ary diagonal operation, and hence
\[ \mathcal{P}(I(A)) = \mathcal{P}(A) = \{1, 2, \ldots\}. \]

PROPOSITION 2. Let \( A \) be an algebra and \( B \) a reduct of \( A \) such that \( B \in \mathcal{A}_{m,n} \) for some \( m \) and \( n \). Then \( \mathcal{P}(I(B)) \) is infinite.

COROLLARY 2. \( \mathcal{P}(A) \) is infinite for all non-one-element algebras \( A \in \mathcal{A}_{m,n} \).

PROBLEM. Is it true that \( \mathcal{P}(I(A)) = \{1, 2, \ldots\} \) for all non-one-element algebras \( A \in \mathcal{A}_{m,n} \)? (P 1083)

If the answer is affirmative, then Marczewski’s conjecture has a positive solution.

THEOREM. If \( A \) is an algebra having bases of different cardinalities, then \( A \) contains infinite independent sets.

Proof of Proposition 1. Let \( A_{1,n} \in \mathcal{A}_{1,n} \); then
\[ A_{1,n} = (A, f, g_1, g_2, \ldots, g_n). \]

Consider now
\[ d(x_1, x_2, \ldots, x_n) = f(g_1(x_1), g_2(x_2), \ldots, g_n(x_n)). \]

Of course, conditions (*) and (**) imply that \( d \) is idempotent, depends on each its variable and its diagonality results from axiom (II) in [5]. Now, it is easy to verify that every operation \( d'(x_1, x_2, \ldots, x_m) = d(x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \) \((1 \leq i_p \leq m \) for \( p = 1, 2, \ldots, n \)) is a diagonal algebraic operation and that \( \mathcal{P}((A, d)) = \{1, 2, \ldots, n\} \). For the last fact see [6]. Now, if \( A \in \mathcal{A}_{m,n} \), then by the Ryll-Nardzewski theorem [2] we have
\[ A^{(n)} = (A^{(n)}, F) \cong A^{(m)} = (A^{(m)}, F) \]

and, therefore, \( A^{(n)} \) has bases whose cardinalities form an arithmetic progression. If \( A \in \mathcal{A}_{1,n} \), then the cardinalities of the bases of \( A^{(1)} = (A^{(1)}, F) \) are of the form \( 1 + (n-1)k \), where \( k = 1, 2, \ldots \) and \( n \) is the smallest cardinality of a basis of \( A \) greater than 1.

Let \( s \leq 1 + (n-1)k = r \) for some \( k \) and let \( \{f_1, f_2, \ldots, f_r\} \) be a basis of \( A^{(r)} \). Then there exists an algebraic operation \( f \) over \( A \) such that
\[ \tilde{f}(f_1, f_2, \ldots, f_r) = e_1^{(r)} \]

and
\[ \tilde{f}(\tilde{f}(e_1^{(r)}, e_2^{(r)}, \ldots, e_r^{(r)})) = e_i^{(r)} \quad \text{for} \ i = 1, 2, \ldots, r. \]
The last formula can be rewritten as follows:

\[ f(f_1(x), f_2(x), \ldots, f_r(x)) = x, \]

\[ f_i(f(x_1, x_2, \ldots, x_r)) = x_i \quad \text{for all } x_1, x_2, \ldots, x_r, x \in A. \]

So far we have shown that the algebra \( \mathfrak{A} \) contains a reduct \( \mathfrak{B} \in \mathcal{A}_{1,r} \). But \( \mathcal{A}(\mathfrak{A}) \supseteq \mathcal{A}(\mathfrak{B}) \) and we can repeat the previous argument for \( \mathfrak{B} \) to prove the existence of an essentially r-ary diagonal operation in the algebra \( \mathfrak{A} \). This completes the proof.

Corollary 1 follows immediately.

Proof of Proposition 2. Let \( \mathfrak{A} = (A, F), F_1 \subset \mathcal{A}(F) \) and \( \mathfrak{B} = (A, F_1) \). Assuming \( \mathfrak{B} \in \mathcal{A}_{n,m} \) we set

\[ \mathfrak{B}^{(m)} \cong (A^{(m)}(F_1), F_1) \cong (A^{(n)}(F_1), F_1) \]

and

\[ \mathcal{S}(I(\mathfrak{B}^{(m)})) \subseteq \mathcal{S}(I(\mathfrak{B}^{(n)})) \subseteq \mathcal{S}(I(\mathfrak{A})). \]

As \( \mathfrak{B}^{(m)} \) has bases of different cardinalities, our assertion follows from Theorem 1 of [2].

The proof of Corollary 2 is trivial.

Proof of the Theorem. Assume that the algebra \( \mathfrak{A} \) has a basis \( B_1 = \{b_1^1, b_2^1, \ldots, b_m^1\} \) of \( m \) elements and also a basis \( B_2 = \{b_1^2, b_2^2, \ldots, b_m^2\} \) of \( m+r \) elements. Now we construct a new basis \( B_3 \) by letting

\[ b_i^1 = g_1(b_1^1, b_2^1, \ldots, b_m^1) \quad \text{for } i = 1, 2, \ldots, m+r \]

and

\[ b_i^2 = b_{m+k}^2 \quad \text{for } i = m+r+k \text{ and } k = 1, 2, \ldots, r. \]

So

\[ B_3 = \{g_1(b_1^1, b_2^1, \ldots, b_m^1), g_2(b_1^1, b_2^1, \ldots, b_m^1), \ldots, g_{m+r}(b_1^1, b_2^1, \ldots, b_m^1), \}
\n
\[ g_{m+r}(b_1^1, b_2^1, \ldots, b_m^1), b_{m+1}^2, b_{m+2}^2, \ldots, b_{m+r}^2\}\]

\[ = \{b_1^3, b_2^3, \ldots, b_{m+2r}^3\}. \]

It is an easy matter to show that \( |B_3| = m+2r \). Let us check that \( B_3 \in \text{Ind}(\mathfrak{A}) \). Assume that for \( F_1, F_2 \in \mathfrak{A}^{(m+2r)} \) the following holds:

\[ F_1(b_1^3, b_2^3, \ldots, b_{m+2r}^3) = F_2(b_1^3, b_2^3, \ldots, b_{m+2r}^3), \]

This means that

\[ F_1(g_1(b_1^3, b_2^3, \ldots, b_m^3), \ldots, g_{m+r}(b_1^3, b_2^3, \ldots, b_m^3), b_{m+1}^2, \ldots, b_{m+r}^2) \]

\[ = F_2(g_1(b_1^3, b_2^3, \ldots, b_m^3), \ldots, g_{m+r}(b_1^3, b_2^3, \ldots, b_m^3), b_{m+1}^2, \ldots, b_{m+r}^2). \]
The last equality is satisfied on the independent set $B_2$, therefore on $A$. Thus by using formula (**) we obtain

$$F_1(x_1, x_2, \ldots, x_{m+2r}) = F_2(x_1, x_2, \ldots, x_{m+2r})$$

for all $x_1, x_2, \ldots, x_{m+2r} \in A$.

This implies $B_2 \in \text{Ind}(\mathbb{A})$. Since $f_j(b_1^2, b_2^2, \ldots, b_{m+r}^2) = b_j^2$ for $j = 1, 2, \ldots, m$, we infer that the subalgebra generated by $B_2$ contains $B_2$. Hence $B_2$ generates $\mathbb{A}$. Let us write

$$J_1 = \{b_{m+1}^2, b_{m+2}^2, \ldots, b_{m+r}^2\},$$

$$J_2 = J_1 \cup \{g_{m+1}(b_1^2, b_2^2, \ldots, b_m^2), \ldots, g_{m+r}(b_1^2, b_2^2, \ldots, b_m^2)\}.$$

Then $|J_1| = r$ and $|J_2| = 2r$ with $J_1 \subset J_2 \in \text{Ind}(\mathbb{A})$. Analogously, starting with the bases $B_2$ and $B_3$ we construct a new base $B_s$ with $|B_s| = m+3r$ and, in general,

$$B_s = \{b_1^s, b_2^s, \ldots, b_{m+(s-1)r}^s\} \quad \text{with} \quad |B_s| = m+(s-1)r.$$

Also, for a given $s$ we construct $J_s$ as follows:

$$J_s = J_{s-1} \cup \{b_{m+r+s}^{s+1}, b_{m+r+1}^{s+1}, \ldots, b_{m+2r}^{s+1}\}.$$

Since $(J_s)$ is an increasing sequence of independent sets, its union is infinite and independent, which completes the proof of the Theorem.

REFERENCES


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