On ultra-weak convergence in $L^p$

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Abstract. Let $(\varphi_n)$ be a sequence in $L^p$ on the unit circle such that

$$\lim_{n \to \infty} \int f(e^{i\theta})\varphi_n(\theta)d\theta = l(f) \quad \text{exists for all } f \in H^q, \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty.$$ 

Then there exists $\varphi \in L^p$ such that

$$l(f) = \int \frac{f(e^{i\theta})\varphi(\theta)d\theta}{2\pi}$$

for all $f \in H^q$. The result is known for $p = 1, q = \infty$, the purpose of this paper is to supply the proofs for the remaining cases.

1. Introduction. Let $\Delta$ denote the unit disk and $T$ its boundary. $L^p$ denotes the usual Lebesgue space considered on $T$ and $H^q$ the Hardy space on $\Delta$. If $f$ is in $H^q$, then $f(e^{i\theta})$, the boundary function of $f$, is considered as an element of $L^p$.

Piranian, Shields and Wells [5] proved the following; which was conjectured by Taylor [6]

**Theorem 1.** Let the sequence $\{a_0, a_1, \ldots\}$ of complex numbers have the property that for each function $\sum b_n z^n$ in $H^\infty$ the limit

$$\lim_{n \to 1} \sum a_n b_n z^n$$

exists and is finite. Then there exists a function $\varphi \in L^1(0, 2\pi)$ such that

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} \varphi(t)e^{int}dt = \hat{\varphi}(n) \quad (n \geq 0).$$

The converse is true. At the end of [5], they conjecture Theorem 2 which if true would imply Theorem 1. Kahane [2] has shown that Theorem 2 is true if $H^\infty$ is replaced by $A$. $A$ denotes the subspace, of $H^\infty$, of functions having continuous boundary values. Mooney [4] then completed the proof of Theorem 2 utilizing Kahane's result.

**Theorem 2.** Let $(\varphi_n) \subset L^1$ such that

$$\lim_{n \to \infty} \int_0^{2\pi} f(e^{i\theta})\varphi_n(\theta)d\theta = l(f)$$
for all \( f \in H^\infty \). Then there exists \( \psi \in L^1 \) such that
\[
 l(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \psi(\theta) d\theta
\]
for all \( f \in H^\infty \).

In this paper we will extend Theorem 2 by replacing \( L^1 \) by \( L^p \) and \( H^\infty \) by \( H^q \), where \( 1/p + 1/q = 1 \), \( 1 \leq p \leq \infty \). Although the method of the proof is similar it is necessary to separate the cases \( 1 < p, q < \infty \) and \( p = \infty, q = 1 \) since \( L^1 \) is not reflexive.

2. Comments on the proof of Theorem 2 and the more general result.

Since \( H^\infty \subset L^1 \), \( 1 < p < \infty \), the hypotheses of Theorem 2 are strengthened if \( L^1 \) is replaced by \( L^p \) and \( H^\infty \) by \( H^q \). Therefore, Theorem 2 still asserts the existence of \( \psi \in L^1 \) such that
\[
 l(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \psi(\theta) d\theta
\]
for \( f \in H^\infty \) if the stronger hypotheses are satisfied. To obtain the stronger conclusion by using Theorem 1 would require two seemingly difficult steps, (1) to show that \( \psi \in L^p \), rather than \( \psi \in L^1 \), (2) to show that the representation is valid for all \( f \in H^q \), instead of just \( H^\infty \). Although Mooney [3] did complete the proof of Theorem 2 by extending the validity of the representation from a subspace to all of \( H^\infty \), this does not seem viable when comparing \( H^\infty \) with \( H^q \). In fact, it is much simpler to proceed directly. However, the attempt to proceed from Theorem 2 makes the general result seem plausible. The case \( p = \infty \) and \( q = 1 \) does not seem to be suggested by Theorem 2.

3. The case \( 1 < p, q < \infty \).

**Theorem 3.** Let \( \{ \psi_n \} \subset L^p \), \( 1 < p < \infty \), such that
\[
 \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \psi_n(\theta) d\theta = l(f)
\]
exists for all \( f \in H^q \), \( 1/p + 1/q = 1 \).

Then there exists \( \psi \in L^p \) such that
\[
 l(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \psi_n(e^{i\theta}) \psi(\theta) d\theta
\]
for all \( f \in H^q \).

**Proof.** Set
\[
 l_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \psi_n(\theta) d\theta;
\]
then \( l_n \in (H^q)^* \) which by the Hahn–Banach Theorem has an extension \( \hat{l}_n \in (L^p)^* \). Moreover, by the Uniform Boundedness Principle the \( l_n \)'s are uniformly bounded and hence the \( \hat{l}_n \)'s. Since \((L^q)^*\) may be identified with \( L^p \) the \( l_n \)'s may be identified with a bounded subset of \( L^p \). Since \( L^p \) is reflexive bounded subsets are weakly compact, there exists \( \varphi \in L^p \) and a subsequence \( \{\varphi_{n_k}\} \subset L^p \) which converges weakly to \( \varphi \), i.e.

\[
\lim_{k \to \infty} \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \varphi_{n_k}(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \varphi(\theta) d\theta
\]

for all \( g \in L^q \). If \( f \in H^q \), then

\[
\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi_{n_k}(\theta) d\theta = \hat{l}_{n_k}(f) = l_{n_k}(f)
\]

but \( \lim_{n \to \infty} l_n(f) = l(f) \) so that

\[
l(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi(\theta) d\theta.
\]

The proof of Theorem 3 is considerably shorter than that of Theorem 2 for several reasons, although it is basically similar. In Kahane's construction it is necessary to restrict \( l_n \) to \( A \) in order to obtain the integral representation for \( l(f) \). Unfortunately the representation is given by a measure so it is then necessary to show that it is absolutely continuous and hence given by an \( L^1 \) function. Because of the restriction to \( A \), Mooney's construction is necessary to show that the representation is valid for \( H^\infty \).

4. The case \( p = \infty, q = 1 \).

**Theorem 4.** Let \( \{\varphi_n\} \subset L^\infty \) such that

\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi_n(\theta) d\theta = l(f)
\]

exists for all \( f \in H^1 \). Then there exists \( \varphi \in L^\infty \) such that

\[
l(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi(\theta) d\theta
\]

exists for all \( f \in H^1 \).

**Proof.** Proceeding as in the proof of Theorem 3 set

\[
l_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) \varphi_n(\theta) d\theta.
\]
Then \( l_n \in (H^1)^* \) and extends to \( \hat{l}_n \in (L^1)^* \). Since \( (L^1)^* \) may be identified with \( L^\infty \), the \( l_n \)'s may be identified with a bounded subset of \( L^\infty \) (use the U.B. Principle again). By Alaoglu’s Theorem, the unit ball of \( L^\infty \) is weak* compact. Without loss of generality we may assume \( \|l_n\| \leq 1 \) so that there is a \( \varphi \in L^\infty \) and a subsequence \( \{\tilde{\varphi}_{nk}\} \) which converges weak* to \( \varphi \), i.e.,

\[
\lim_{k \to \infty} \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\tilde{\varphi}_{nk}(\theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\varphi(\theta) d\theta
\]

for all \( g \in L^1 \), where \( \tilde{\varphi}_{nk} \) is identified with \( \hat{l}_{nk} \) which in the extension of \( l_n \). For \( f \in H^1 \)

\[
\frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\tilde{\varphi}_{nk}(\theta) d\theta = \hat{l}_{nk}(f) = \tilde{l}_{nk}(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\varphi_{nk}(\theta) d\theta
\]

and by hypothesis

\[
\lim_{n \to \infty} \tilde{l}_n(f) = l(f)
\]

so

\[
\lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\varphi_{nk}(\theta) d\theta = l(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})\varphi(\theta) d\theta.
\]

Combining Theorems 2, 3, 4 gives the desired complete general result.

4. Results. E. A. Heard [1] has announced a new proof of the weak sequential completeness of \( L^1 \) using Kahane’s results. A similar approach to the weak sequential completeness of \( L^p \), \( 1 < p < \infty \), is of no consequence, however, since the reflexive property was used in the proof of Theorem 3.

Unlike Kahane’s and Mooney’s results, Theorems 3 and 4 are independent of dimension, that is both generalize to \( \Delta^N \) and \( T^N \) without any change in the proofs as follows.

**Theorem 5.** Let \( \Delta^N \) and \( T^N \) denote the \( N \)-dimensional polydisc in \( C^N \) and its distinguished boundary, \( N \geq 1 \). Let \( 1 < p \leq \infty \), \( 1/p + 1/q = 1 \). If \( \{\varphi_n\} \subset L^p(T^N) \) such that

\[
\lim_{n \to \infty} \tilde{l}_n(f) = \lim_{n \to \infty} \int_{T^N} f^* \varphi_n = l(f)
\]

exists for all \( f \in H^q(\Delta^N) \) (\( f^* \) is the boundary function of \( f \)), then there exists \( \varphi \in L^p(T^N) \) such that

\[
l(f) = \int_{T^N} f^* \varphi
\]

for all \( f \in H^q(\Delta^N) \).
As observed by Kahane the hypothesis of Theorem 2 is the existence of
\[ \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{n,k} b_k, \] where \( \varphi_n(\theta) = \sum_{k=-\infty}^{\infty} a_{n,k} e^{-ik\theta}, \) for all \( \sum_{k=0}^{\infty} b_k e^{ik\theta} \in H^\infty(\Delta). \)

The conclusion of the theorem is \( \lim_{n \to \infty} a_{n,k} = \int \varphi(\theta) e^{ik\theta} d\theta \) for some \( \varphi \in L^1(\mathbb{T}). \) Theorem 1 is a special case of this re-statement. In like manner Theorems 3 and 4 can be re-stated to give results analogous to Theorem 1.

**An example.** In the proofs of Theorems 2, 3, 4, 5 a crucial step is the extraction of a convergent subsequence. This convergent subsequence is obtained by the weak* sequential compactness of the unit ball, rather than just weak* compactness; and separability is a sufficient condition. The following example found in [3], p. 311, shows that in general separability can not be omitted.

By the natural imbedding \( l^1 \) is a subspace of \( (l^\infty)^* \) then \( \{e_k\} \subset l^1 \) is a bounded sequence in \( (l^\infty)^* \) but no subsequence is weakly convergent in \( (l^\infty)^* \).

**References**


*Reçu par la Rédaction le 25. 2. 1974*