ON SMALL STOCHASTIC PERTURBATIONS
OF MAPPINGS OF THE UNIT INTERVAL

BY

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INTRODUCTION

We consider small stochastic perturbations of mappings from the unit interval into itself (for more detailed information see [3]). The general setting of the problem is as follows. Let \( I = [0, 1] \) and let \( \tau: I \to I \) be a piecewise monotonic mapping of class \( C^1 \), i.e. there exist \( 0 = b_0 < b_1 < \ldots < b_{q-1} < b_q = 1 \) such that, for \( i = 1, 2, \ldots, q \), \( \tau_{[b_{i-1}, b_i]} \) is a monotonic function of class \( C^1 \) and, moreover, it can be extended to \( [b_{i-1}, b_i] \) as a \( C^1 \)-function, which will be denoted by \( \tau_i \).

Let \( m \) be the Lebesgue measure on \( I \) and let \( L^1 = L^1(I, m) \) be the space of all \( m \)-integrable real functions on \( I \). We denote by \( L^1_+ \) the subset of \( L^1 \) containing all nonnegative functions \( f \) satisfying the condition \( \int f(x) \, dx = 1 \).

With the mapping \( \tau \) one can associate the Perron-Frobenius operator \( P_\tau: L^1 \to L^1 \) so that

\[
(P_\tau f)(y) = \sum_{i=1}^{q} \frac{f(\tau_i^{-1} y)}{|\tau_i'(\tau_i^{-1} y)|}, \quad y \in I,
\]

where \( f(\tau_i^{-1} y) = 0 \) for \( y \notin \tau_i([b_{i-1}, b_i]) \). It is well known that \( P_\tau(L^1_+) \subseteq L^1_+ \) and \( P_\tau f = f \) if and only if the measure \( f m \) is invariant under \( \tau \). Let \( L(\tau) \) denote the set of functions in \( L^1_+ \) invariant under \( P_\tau \).

For any positive integer \( n \), we consider a family of probability densities \( q^n(x, \cdot) \), \( x \in I \), with respect to the measure \( m \). The densities \( q^n \) considered below are bounded and measurable as functions of two variables. The family of transition densities \( p^n(x, \cdot) = q^n(\tau(x), \cdot) \), \( n = 1, 2, \ldots \), with respect to \( m \) is called a stochastic perturbation of the mapping \( \tau \). It is called small if for any \( r > 0 \) we have

\[
\inf_{x \in I, x \neq \tau^{-r}} \int_{x-r}^{x+r} q^n(x, y) \, dy \to 1 \quad \text{as} \quad n \to \infty.
\]

Perturbations considered in the sequel are small as they are local, i.e. for
$n = 1, 2, \ldots$ there exists $r_n > 0$ such that $q^n(x, y) = 0$ for $|y - x| > r_n$, and $r_n \to 0$ as $n \to \infty$.

We define operators $Q_n$ and $P_n$, $n = 1, 2, \ldots$, from $L^1$ into itself as follows:

$$(Q_n f) (y) = \int q^n(x, y) f(x) \, dx, \quad (P_n f) (y) = \int p^n(x, y) f(x) \, dx, \quad y \in I.$$  

Here and throughout the paper we neglect the indication of the range of integration if that range is the interval $I$. It is easy to see that $P_n = Q_n \circ P$, $n = 1, 2, \ldots$

Under our assumptions the transition density $p^n$ ($n = 1, 2, \ldots$) has at least one invariant probability measure $\mu_n$, i.e.,

$$\mu_n(A) = \int \int p^n(x, y) \, dy \, d\mu_n(x)$$

for any Borel subset $A$ of $I$. The measure $\mu_n$ is of the form $\mu_n = f_n \, m$, where $f_n \in L^1$, and $P_n f_n = f_n$ (see [2]).

Our aim is to find the limit points of the set $\{\mu_n; n = 1, 2, \ldots\}$ in the weak topology of measures ($\mu_n \to \mu \iff \mu_n(g) \to \mu(g)$ for any continuous function $g$: $I \to R$). Any such limit point will be called the limit measure for the perturbation $p^n$ ($n = 1, 2, \ldots$).

In the paper, we discuss mappings $\tau$ from 3 different classes. In part I, $\tau$ is a piecewise monotonic mapping of class $C^2$ as in [4] and [5]. Our results can be easily generalized to mappings $\tau$ that are piecewise monotonic and of class $C^1$ with $|1/\tau'|$ of bounded variation. Such mappings have been considered by Wong [11]. In part II, $\tau$ is a piecewise monotonic mapping of class $C^{1+\epsilon}$ as in [9].

The perturbations we consider are of two classes. Their definitions are given in Sections I.A and I.B, respectively. Perturbations of Section I.A are connected with one of Ulam's problems [10] (see Example I.A.1).

The main result of the paper is the proof of the theorem that under our assumptions the limit measures for small stochastic perturbations are of the form $fm$, where $f \in L(\tau)$. This result may be understood as a stability of absolutely continuous $\tau$-invariant measures under some classes of small stochastic perturbations.

In the paper we use the methods analogous to those of Li [6].

The author is much indebted to K. Krzyżewski for suggesting the subject and many inspiring talks.

1. PIECEWISE $C^2$-MAPPINGS

In this part of the paper, $\tau$ is a piecewise monotonic mapping of class $C^2$, i.e. for $i = 1, 2, \ldots, q$ the function $\tau_i$ is monotonic and of class $C^2$.

We shall use the following lemma from [5]:
Lemma 1. For any \( f \in L_+^1 \) and
\[
K = (\sup |\tau'|)(\inf |\tau'|)^{-2} + 2(\inf |\tau'|)^{-1}(\min_{1 \leq i \leq q} (b_i - b_{i-1}))^{-1}
\]
we have
\[
V_0^1 (P_\tau f) \leq 2(\inf |\tau'|)^{-1} V_0^1 (f) + K,
\]
where \( V_0^b(g) \) is the variation of the function \( g \) on the interval \([a, b] \).

I.A. "Average-like" perturbations. In this section we claim that \(|\tau'| > 2\) for \( i = 1, 2, \ldots, q \).

Let \( \pi_n = \{I_{n,1}, \ldots, I_{n,m(n)}\} \) be a partition of \( I \) into closed intervals such that \( I_{n,i-1} \cap I_{n,i} \) is a single point \((n = 1, 2, \ldots \) and \( i = 1, 2, \ldots, m(n) \)). We claim that
\[
\max \{m(I_{n,i}) : i = 1, 2, \ldots, m(n)\} \to 0 \quad \text{as } n \to \infty.
\]
Let us define
\[
q(\pi_n)(x, y) = \begin{cases} (m(I_{n,i}))^{-1} & \text{for } x, y \in I_{n,i}, \\ 0 & \text{otherwise}. \end{cases}
\]
The definitions of \( p(\pi_n), Q(\pi_n), \) and \( P(\pi_n) \) are analogous to those of \( p_n, Q_n, \) and \( P_n \) in the Introduction \((n = 1, 2, \ldots)\).

Since \( Q(\pi_n) \) is an operator of conditional expectation, we call the perturbations generated by \( q(\pi_n) \)'s average-like perturbations.

The main technical result of this section is the following

Proposition I.A. Let \( f_n \) belong to \( L_+^1 \) and let \( P(\pi_n) f_n = f_n \) for \( n = 1, 2, \ldots \). Then the set \( \{f_n : n = 1, 2, \ldots\} \) is relatively compact in \( L^1 \) and its limit points belong to \( L(\tau) \).

The proof of Proposition I.A is based on two lemmas.

Lemma I.A.1. For any positive integer \( n \) and for any \( f \in L^1 \) we have
\[
V_0^1 (Q(\pi_n) f) \leq V_0^1 (f).
\]

Lemma I.A.2. For any \( f \in L^1 \) we have \( Q(\pi_n) f \to f \) as \( n \to \infty \) in the \( L^1 \)-norm. The convergence is uniform on relatively compact subsets of \( L^1 \).

The proofs of Lemmas I.A.1 and I.A.2 are analogous to the proofs of the appropriate lemmas in [6].

Proof of Proposition I.A. Let \( f_n \in L_+^1 \) be invariant for \( P(\pi_n) \), where \( n = 1, 2, \ldots \). Since \( P(\pi_n) = Q(\pi_n) \circ P_\tau \), by Lemmas I and I.A.1 we have
\[
V_0^1 (f_n) = V_0^1 (P(\pi_n) f_n) = V_0^1 (Q(\pi_n)(P_\tau f_n)) \leq V_0^1 (P_\tau f_n) \leq 2(\inf |\tau'|)^{-1} V_0^1 (f_n) + K.
\]
Hence \( V_0^1 (f_n) \leq K (1 - 2(\inf |\tau'|)^{-1})^{-1} \) for \( n = 1, 2, \ldots \). Since \( ||f_n||_{L^1} = 1 \) for any positive integer \( n \), we infer that \( ||f_n||_{L^\tau} \) \((n = 1, 2, \ldots)\) are uniformly bounded.
Applying Helly’s theorem [8] we see that the set \( \{f_n: n = 1, 2, \ldots \} \) is relatively compact in \( L^1 \).

Let \( f_{n_i} (i = 1, 2, \ldots) \) be a subsequence converging to a function \( f \) in \( L^1 \) as \( i \to \infty \). We prove that \( f \in L(\tau) \). We have

\[
\|P_\tau f - f\|_1 \leq \|P_\tau f - P_\tau f_{n_i}\|_1 + \|P_\tau f_{n_i} - Q(\pi_{n_i}) P_\tau f_{n_i}\|_1 + \\
+ \|Q(\pi_{n_i}) P_\tau f_{n_i} - f_{n_i}\|_1 + \|f_{n_i} - f\|_1
\]

with all summands on the right-hand side being arbitrarily small. Thus \( P_\tau f = f \), which completes the proof.

The main result of this section is the following theorem, which is an immediate corollary to Proposition I.A.

**Theorem I.A.** If, for any positive integer \( n \), \( \mu_n \) is a probability Borel measure invariant for \( p(\pi_n) \), then the limit points of the set \( \{\mu_n: n = 1, 2, \ldots\} \), in the weak topology, are of the form \( \mu f, f \in L(\tau) \). Moreover, the convergence is in the total variation norm.

**Remark I.A.** For mappings \( \tau \) considered in this part of the paper Kosjakin and Sandler [4] and Li and Yorke [7] have proved that the set of ergodic probability measures is finite and the support of any such measure is a union of a finite number of intervals. Hence any absolutely continuous \( \tau \)-invariant measure can be obtained as a limit measure for a perturbation \( p(\pi_n), n = 1, 2, \ldots \) It is enough to make a suitable choice of the sequence of partitions \( \pi_n \).

**Example I.A.1.** Choose \( \pi_n^0 = \{I_{n,1}, \ldots, I_{n,n}\} \) with \( I_{n,i} = [i-1/n, i/n] \), where \( i = 1, 2, \ldots, n \), and \( n = 1, 2, \ldots \) Ulam [10] has defined an operator \( P_n(\tau): D_n \to D_n \) with \( D_n = \text{Span} \{\chi_{n,i}: i = 1, 2, \ldots, n\} \) in \( L^1 \) and \( \chi_{n,i} \) the characteristic function of the interval \( I_{n,i} \) as follows:

\[
P_n(\tau)(\chi_{n,i}) = \sum_{j=1}^n P_{ij} \chi_{n,j},
\]

where

\[
P_{ij} = \frac{m(I_{n,i} \cap (I_{n,j})^{-1})}{m(I_{n,i})}, \quad 1 \leq i, j \leq n.
\]

Ulam has conjectured that if, for any positive integer \( n \), \( f_n \in D_n \) is invariant for \( P_n(\tau) \), then the \( L^1 \) limit points of the set \( \{f_n: n = 1, 2, \ldots\} \) belong to \( L(\tau) \). Li has answered the conjecture positively for \( \tau \) discussed in this section (see [6]). It is easy to check that for any positive integer \( n \) the operator \( P(\pi_n^0) \) is an extension of \( P_n(\tau) \) to the whole \( L^1 \). Moreover, \( P(\pi_n^0)(L) \subset D_n \), and so \( P(\pi_n^0) f = f \) implies \( P_n(\tau) f = f \). Hence Proposition I.A is a generalization of Li’s result.
Example I.A.2 (see [1]). The example shows that for mappings \( \tau \) in a very special class one can obtain an absolutely continuous invariant measure for \( \tau \) directly as an invariant measure for a stochastic perturbation.

Let \( \tau \) be a mapping of the unit interval \( I \) into itself for which there exist \( 0 = b_0 < b_1 \leq \ldots \leq b_{q-1} < b_q = 1 \) such that \( \tau_{|\{b_{i-1}, b_i\}} \) is a linear function, \( \tau_{\{b_0, b_1, \ldots, b_q\}} \subset \{b_0, b_1, \ldots, b_q\} \), and \( \tau \neq 0 \). Let \( \pi = \{I_i; \ i = 1, 2, \ldots, q\} \) denote the partition of \( I \) into intervals \( I_i = [b_{i-1}, b_i] \) and let \( q(\pi), p(\pi), P(\pi) \) be defined as above. The set

\[
D = \{ \sum_{i=1}^{q} \alpha_i \chi_i; \sum_{i=1}^{q} \alpha_i m(I_i) = 1 \}
\]

is compact and convex in \( L^1 \) (\( \chi_i \) is the characteristic function of \( I_i \)). For \( y \in I \) we have

\[
(P(\pi) \chi_i)(y) = \int \chi_i(x) p(\pi)(x, y) \, dx = \sum_{j: \tau_{I_i} \supset I_j} \frac{m(I_i \cap \tau^{-1}(I_j))}{m(I_j)} \chi_j(y)
\]

\[
= \frac{1}{|\tau_i|} \chi_{\tau_{I_i}}(y) = P(\pi)(y), \quad i = 1, 2, \ldots, q,
\]

so \( P(\pi)_D = P_{|D} \) and \( P(\pi)(D) \subset D \). Hence there exists a piecewise constant function \( f \in D \) such that \( P_{\tau} f = P(\pi)f = f \).

I.B. "Convolution-like" perturbations. In this section we claim that \( |\tau_i| > 4 \) for \( i = 1, 2, \ldots, q \).

Fix the sequence of positive numbers \( r_n \ (n = 1, 2, \ldots) \), monotonically tending to zero, \( r_1 < 1/4 \). Let \( s^n: R \to R^+ \ (n = 1, 2, \ldots) \) be an \( m \)-measurable bounded function satisfying the following conditions:

(i) \( s^n(t) = 0 \) for \( |t| > r_n \),

(ii) \( s^n(-t) = s^n(t) \),

(iii) \( \int_{-r_n}^{r_n} s^n(t) \, dt = 1 \).

We define a family of probability densities \( q^n \ (n = 1, 2, \ldots) \) with respect to the Lebesgue measure \( m \) as follows:

\[
q^n(x, y) = \begin{cases} 
   s^n(y-x) & \text{for } x \in [r_n, 1-r_n], \\
   s^n(y-x) + s^n(\overline{y}-x) & \text{for the remaining } x \in I,
\end{cases}
\]

where \( \overline{y} = -y \) for \( y \in [0, 1/4] \) and \( \overline{y} = 1+(1-y) \) for \( y \in [3/4, 1] \). Let \( p^n, Q_n, \) and \( P^n \) be defined as in the Introduction.

The perturbations generated by the probability densities \( q^n \) are similar at all points of \( I \) (except for the points near the ends of \( I \)). We call them convolution-like perturbations (see Lemma I.B.1).

We shall prove the following proposition analogous to Proposition I.A.
**Proposition I.B.** Let \( f_n \) belong to \( L^1_+ \) and let \( P_n f_n = f_n \) for \( n = 1, 2, \ldots \). Then the set \( \{ f_n : n = 1, 2, \ldots \} \) is relatively compact in \( L^1 \) and its limit points belong to \( L(\tau) \).

Before proving Proposition I.B, we give the following definition and some lemmas.

For any \( f : I \to R \) we define its extension \( \bar{f} : R \to R \) as follows:

\[
\bar{f}(x) = \begin{cases} 
  f(-x) & \text{for } x \in [\frac{1}{4}, 0), \\
  f(x) & \text{for } x \in I, \\
  f(1 - (x - 1)) & \text{for } x \in (1, \frac{5}{4}], \\
  0 & \text{for the remaining } x \in R.
\end{cases}
\]

**Lemma I.B.1.** For any positive integer \( n \) and for any function \( f \in L^1 \) we have

\[
(Q_n f)(y) = \int_{\frac{1}{4} + r_n}^{y + r_n} \bar{f}(x) s^n(y - x) \, dx = (\tilde{f} * s^n)(y), \quad y \in I.
\]

**Proof.** Let \( y \in [0, r_n]. \) We have

\[
(Q_n f)(y) = \int_{0}^{y + r_n} f(x) q^n(x, y) \, dx
\]

\[
= \int_{0}^{y + r_n} f(x) s^n(y - x) \, dx + \int_{0}^{y + r_n} f(x) s^n(\tilde{y} - x) \, dx
\]

\[
= \int_{0}^{y + r_n} f(x) s^n(y - x) \, dx + \int_{y - r_n}^{y + r_n} \bar{f}(x) s^n(y - x) \, dx = \int_{y - r_n}^{y + r_n} \bar{f}(x) s^n(y - x) \, dx.
\]

The proof for \( y \in [1 - r_n, 1] \) is analogous and for \( y \in (r_n, 1 - r_n) \) it is trivial.

**Lemma I.B.2.** For any \( f \in L^1_+ \) and for any positive integer \( n \) we have \( V^1_0 (Q_n f) \leq 2V^1_0 (f) \).

**Proof.** By Lemma I.B.1, \( Q_n f = \tilde{f} * s^n \). Fix a positive integer \( N \) and a sequence \( 0 = t_0 < t_1 < \ldots < t_N = 1; \) we then have

\[
\sum_{i=1}^{N} |(Q_n f)(t_i) - (Q_n f)(t_{i-1})| = \sum_{i=1}^{N} |(\tilde{f} * s^n)(t_i) - (\tilde{f} * s^n)(t_{i-1})|
\]

\[
= \sum_{i=1}^{N} (s^n * \tilde{f})(t_i) - (s^n * \tilde{f})(t_{i-1})
\]

\[
= \sum_{i=1}^{N} \left| \int_{-r_n}^{r_n} s^n(t) \tilde{f}(t_i - t) \, dt - \int_{-r_n}^{r_n} s^n(t) \tilde{f}(t_{i-1} - t) \, dt \right|
\]

\[
\leq \int_{-r_n}^{r_n} \left( \sum_{i=1}^{N} |\tilde{f}(t_i - t) - \tilde{f}(t_{i-1} - t)| \right) s^n(t) \, dt
\]

\[
\leq \int_{-r_n}^{r_n} V^1_{-r_n} (\tilde{f}) s^n(t) \, dt = V^1_{-r_n} (\tilde{f}) \leq 2V^1_0 (f).
\]
**Lemma I.B.3.** For any \( f \in L^1 \) we have \( Q_n f \to f \) as \( n \to \infty \) in the \( L^1 \)-norm. The convergence is uniform on relatively compact subsets of \( L^1 \).

**Proof.** Since for any positive integer \( n \) the operator norm of \( Q_n \) is equal to 1 and since continuous functions are dense in \( L^1 \), it is enough to prove that \( Q_n g \to g \) in \( L^1 \) as \( n \to \infty \) for any continuous function \( g \).

We first prove that for any \( y \in I \)

\[
\int q^n(x, y) \, dx = 1, \quad n = 1, 2, \ldots
\]

Let \( y \in [0, r_n] \); we have

\[
\int q^n(x, y) \, dx = \int_0^{r_n} (s^n(y-x) + s^n(y-x)) \, dx + \int_{y-r_n}^{y+r_n} s^n(y-x) \, dx.
\]

Since

\[
\int_0^{r_n} s^n(y-x) \, dx = \int_0^{y-r_n} s^n(y-x) \, dx = \int_{y-r_n}^{y+r_n} s^n(y-x) \, dx,
\]

we have

\[
\int q^n(x, y) \, dx = \int_{y-r_n}^{y+r_n} s^n(y-x) \, dx = 1,
\]

as desired. For \( y \in [1-r_n, 1] \) the proof is analogous and for \( y \in (r_n, 1-r_n) \) it is trivial.

Hence

\[
\int |g(y) - (Q_n g)(y)| \, dy = \int |g(y) - \int g(x) q^n(x, y) \, dx| \, dy
\]

\[
\leq \int \left| \int g(y) - g(x) q^n(x, y) \, dx \right| \, dy \leq \int \omega(r_n) q^n(x, y) \, dx \, dy = \omega(r_n),
\]

where \( \omega \) is the modulus of continuity of the function \( g \) and \( \omega(r_n) \to 0 \) as \( n \to \infty \).

We are now in a position to prove Proposition I.B. It is implied by Lemmas I.B.2, I.B.3, and Lemma 1 in the same way as Proposition I.A follows from Lemmas I.A.1, I.A.2, and Lemma I.

The main result of this section is the following

**Theorem I.B.** If, for any positive integer \( n \), \( \mu_n \) is a probability Borel measure invariant for \( p^n \), then the limit points of the set \( \{\mu_n: n = 1, 2, \ldots\} \), in the weak topology, are of the form \( fm, f \in L(\tau) \). Moreover, the convergence is in the total variation norm.

This theorem is a direct consequence of Proposition I.B.

**Remark I.B.1.** The assumption (i) can be replaced by a weaker one:

(i') \( s^n(t) = 0 \) for \( |t| > 1/4 \), \( n = 1, 2, \ldots \), and for any \( r > 0 \)

\[
\int_{-r}^{r} s^n(t) \, dt \to 1 \quad \text{as} \quad n \to \infty.
\]

The proof remains almost unchanged.
Remark I.B.2. Contrary to the situation discussed in Section I.A, there are absolutely continuous $\tau$-invariant measures, which cannot be obtained as limit measures for perturbations considered here. One can formulate the following sufficient conditions for obtaining a measure $m_1 \ll m$ as the limit measure for a perturbation $p^n$ ($n = 1, 2, \ldots$):

(a) $m_1$ is $\tau$-ergodic;

(b) there exists an open set $U \ni \sup m_1$ such that every absolutely continuous $\tau$-ergodic measure $m_2$ different from $m_1$ vanishes on $U$;

(c) $\text{cl}(\tau U) = U$, i.e. supp $m_1$ is an "attractor".

For the proof note that, by (c), for $n$ large enough there exists a measure $\mu_n$ invariant for $p^n$ and concentrated in $U$. We know that any limit point of the set $\{\mu_n: n = 1, 2, \ldots\}$ (in the weak topology) is an absolutely continuous invariant measure for $\tau$. From (a) and (b) it follows that any such limit point equals $m_1$.

We believe that condition (c) is necessary for obtaining $m_1$ as a limit measure, but we have no proof of that.

To illustrate the above remarks, we give two examples (see Figs. 1 and 2).

![Fig. 1](image1)

![Fig. 2](image2)

For $\tau$ of Fig. 1 there are two ergodic absolutely continuous probability measures: $m_1 = 2m_{[0,1/2]}$ and $m_2 = 2m_{[1/2,1]}$. For any positive integer $n$ the only invariant probability measure for $p^n$ is $m = \frac{1}{2}m_1 + \frac{1}{2}m_2$.

For $\tau$ of Fig. 2 there are two ergodic absolutely continuous probability measures: $m_1 = 2m_{[0,1/2]}$ and $m_2$ with support in $[1/2+\delta, 1]$. For any $n$ such that $r_n < \delta/2$ there exists only one invariant probability measure $\mu_n$ for the transition density $p^n$. Since $\mu_n ([0, 1/2]) = 0$, we have $\mu_n \to m_2$ ($n \to \infty$) in the weak topology.

Remark I.B.3. The results of Sections I.A and I.B remain true for $\tau$ considered by Wong [11], i.e. $\tau$ piecewise monotonic and of class $C^1$ with
$|1/\tau'|$ of bounded variation, if we claim in addition that $\inf |\tau'| > 3$ in Section I.A or $\inf |\tau'| > 6$ in Section I.B. The only change in proofs is replacing Lemma I by an analogous one from [11].

II. PIECEWISE $C^{1+\varepsilon}$-MAPPINGS

In this part we discuss small stochastic perturbations of piecewise monotonic expanding mappings $\tau$ of class $C^{1+\varepsilon}$, i.e. for any $i = 1, 2, \ldots, q$ the mapping $\tau_i$ satisfies the Hölder condition with a constant $\alpha$ and an exponent $\varepsilon$. We claim that $\inf |\tau_i| \geq \lambda > 1$, $i = 1, 2, \ldots, q$. The existence of absolutely continuous invariant measures for such mappings has been proved by Wong [12] under some additional very restrictive conditions and, recently, without any supplementary assumptions by Rychlik [9].

For functions from $L^1$ Rychlik has introduced a quantity $C_{\varepsilon}$ which for $\tau$ piecewise of class $C^{1+\varepsilon}$ plays an analogous role to that of the variation $V_0^1$ for $\tau$ piecewise of class $C^2$.

It is worth noting that Rychlik’s method applies to expanding mappings $\tau$ for which the modulus of continuity $\omega$ of $\tau_i$ ($i = 1, 2, \ldots, q$) satisfies the condition

$$\sup_{\delta > 0} \frac{\omega(\delta/\lambda)}{\omega(\delta)} < 1.$$  

The results of this part can be easily generalized to small stochastic perturbations of such mappings.

Now we claim additionally that $(\lambda)^{-\varepsilon} + 4(\lambda)^{-1} < 1$. Let us denote by $a_j$ ($j = 1, 2, \ldots, \bar{q}$) all different points $\tau_i(b_{i-1}), \tau_i(b_i)$ ($i = 1, 2, \ldots, q$). Let $\delta_0 > 0$ satisfy the following conditions:

(i) $(\lambda)^{-\varepsilon} + 4(\lambda)^{-1} + \alpha(\delta_0)^{\varepsilon}(\lambda)^{-1-2\varepsilon} < 1$;
(ii) intervals $[a_j-\delta_0, a_j+\delta_0]$ for $j = 1, 2, \ldots, \bar{q}$ are disjoint;
(iii) $\delta_0 < \frac{1}{3} \min_{1 \leq i \leq q} (b_i-b_{i-1})$.

For $f \in L^1$ we write

$$A(f, \delta, x) = \sup_{|x-y| < \delta} |f(x) - f(y)|$$

and

$$C_{\varepsilon}(f) = \sup_{0 < \delta \leq \delta_0} (\delta)^{-\varepsilon} \int A(f, \delta, x) dx.$$  

Then $C_{\varepsilon}$ is a seminorm on the subspace of $L^1$ composed of all functions $f$ such that $C_{\varepsilon}(f) < \infty$. Any subset of $L^1_+$ bounded in the seminorm $C_{\varepsilon}$ is relatively compact in $L^1$. We shall prove this for a countable set $\{f_n \in L^1_+: n = 1, 2, \ldots\}$.  

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If \( \omega_n \) is the integral modulus of continuity of the function \( f_n \), then for \( \delta \leq \delta_0/2 \) we have

\[
\omega_n(\delta) = \int |f_n(\delta + x) - f_n(x)| \, dx \leq \int A(f_n, 2\delta, x) \, dx \leq \bar{C}_e \cdot 2^q \delta^q,
\]

where \( \bar{C}_e = \sup \{ C_e(f_n): n = 1, 2, \ldots \} \). Thus the functions \( f_n \) (\( n = 1, 2, \ldots \)), uniformly bounded in the \( L \)-norm, have also a common integral modulus of continuity. Hence they form a relatively compact set in \( L^1 \).

Now, we recall two lemmas from [9].

**Lemma II.1.** For any \( f \in L^1 \) we have

\[
C_e(P_\xi f) \leq (\lambda^{-\xi} + 4\lambda^{-1} + \alpha \delta_0 \lambda^{-1-2\xi}) C_e(f) + (\alpha \lambda^{-1-\xi} + 2\lambda^{-1} \delta_0^{-\xi}) \| f \|_{L^1}.
\]

**Lemma II.2.** If \( f \in L^1 \) and \( x, y \) belong to an interval \( J \) such that \( m(J) = k\delta \) for some positive integer \( k \), then

\[
|f(x) - f(y)| \leq (2/\delta) \int_J A(f, \delta, x) \, dx.
\]

**II.A. "Average-like" perturbations.** In this section, for \( \tau \) as above we consider small stochastic perturbations of the type discussed in Section I.A.

Let \( \pi_n \), \( q(\pi_n) \), \( p(\pi_n) \), \( Q(\pi_n) \), and \( P(\pi_n) \) be as in Section I.A.

**Lemma II.A.1.** For any \( f \in L^1_+ \) and for any positive integer \( n \) we have

\[
C_e(Q(\pi_n) f) \leq 18 C_e(f).
\]

**Proof.** Fix a function \( f \in L^1_+ \) and a positive integer \( n \). We put \( g = Q(\pi_n) f \). The function \( g \) is constant on elements of the partition \( \pi_n \). On any such element the value of \( g \) is equal to the mean value of the function \( f \) on this interval.

Fix \( \delta \leq \delta_0 \) and consider intervals \( I_i = [(i-1)\delta, i\delta] \) for \( i = 1, 2, \ldots, w \) and \( I_{w+1} = [w\delta, 1] \), where \( w = E(\delta^{-1}) \). Moreover, assume that \( I_i = \emptyset \) for \( i \notin \{1, 2, \ldots, w+1\} \).

Let

\[
O(h, J) = \sup_J h - \inf_J h
\]

for any interval \( J \) and any function \( h \) from \( L^1_+ \). For \( x \in I_i, i = 1, 2, \ldots, w+1 \), we have

\[
A(g, \delta, x) \leq O(g, I_{i-1} \cup I_i \cup I_{i+1}).
\]

Thus

\[
\int A(g, \delta, x) \, dx \leq \delta \sum_{i=1}^{w+1} O(g, I_{i-1} \cup I_i \cup I_{i+1}).
\]

The sets \( I_{i-1} \cup I_i \cup I_{i+1} \) (\( i = 1, 2, \ldots, w+1 \)) cover the interval \( I = [0, 1] \) at
most three times. They form three families, say $F_1, F_2, F_3$, that cover or almost cover $I$. We consider the first of them. Let $F_1$ be the family of intervals

$$J_j = [(2 + 3(j-1))\delta, (2 + 3j)\delta] \cap I, \quad j = 0, 1, \ldots, \tilde{w}.$$  

We assume the worst possibility, i.e. that $F_1$ is a covering of $I$ ($\tilde{w} = (w-1)/3$ is a positive integer). Let $j_1 < j_2 < \ldots < j_k, j_i \in \{0, 1, \ldots, \tilde{w}\}, i = 1, 2, \ldots, k,$ be all indices $j$ such that $j = 0$ or $j = \tilde{w}$ or the function $g$ is not continuous on the interval $J_j$. Of course, we have

$$\sum_{j=0}^{\tilde{w}} O(g, J_j) = \sum_{j=j_1,\ldots,j_k} O(g, J_j).$$

Define sets

$$A_i = \bigcup_{j_{i-1} < j \leq j_i+1} J_j \quad \text{for } i = 1, 2, \ldots, k.$$  

There exist points $x_i, y_i \in A_i$ ($i = 1, 2, \ldots, k$) such that

$$f(x_i) \leq \inf_{J_{j_i}} g \leq \sup_{J_{j_i}} g \leq f(y_i).$$

Thus $O(g, J_{j_i}) \leq O(f, A_i), i = 1, 2, \ldots, k$. Using Lemma II.2 we obtain the estimate

$$O(g, J_{j_i}) \leq O(f, A_i) \leq (2/\delta) \int_{A_i} A(f, \delta, x) dx$$

for $i = 1, 2, \ldots, k-1$. For $i = k$ there are three possibilities:

(i) $x_k, y_k \in A_k \cap [0, w\delta]$; then the estimate analogous to the above one is true;

(ii) exactly one of the points $x_k, y_k$ (say $x_k$) belongs to the interval $[w\delta, 1]$; then

$$O(g, J_{j_k}) \leq |f(x_k) - f(y_k)| \leq |f(x_k) - f(w\delta)| + |f(w\delta) - f(y_k)|$$

$$\leq (2/\delta) \int_{1-\delta}^{1} A(f, \delta, x) dx + (2/\delta) \int_{A_k \cap [0,w\delta]} A(f, \delta, x) dx;$$

(iii) both of the points $x_k, y_k$ belong to $[w\delta, 1]$; then

$$O(g, J_{j_k}) \leq (2/\delta) \int_{1-\delta}^{1} A(f, \delta, x) dx + (2/\delta) \int_{1-\delta}^{1} A(f, \delta, x) dx.$$  

In any of these cases, the sets we integrate over cover the interval $I$ at most three times. Hence

$$\sum_{j=0}^{\tilde{w}} O(g, J_j) \leq 3(2/\delta) \int A(f, \delta, x) dx.$$
Since for $F_2$ and $F_3$ the analogous estimates are true, we obtain
\[ \int A(g, \delta, x) \, dx \leq 3 \cdot 3 \cdot 2 \int A(f, \delta, x) \, dx. \]

Hence $C_\varepsilon(g) \leq 18 C_\varepsilon(f)$, which completes the proof.

Lemmas II.A.1, II.1, and I.A.2 imply the following

**Theorem II.A.** Let $\tau$ be as described above. If $18(\lambda^{-\varepsilon} + 4\lambda^{-1}) < 1$, then the analogues of Proposition I.A and Theorem I.A are true.

**II.B. “Convolution-like” perturbations.** In this section we discuss, for $\tau$ as above, small stochastic perturbations of the type considered in Section I.B. Let $r_n$, $s^n$, $q^n$, $p^n$, $Q_n$, and $P_n$ be as in Section I.B.

**Lemma II.B.1.** For any $f \in L_+$ and for any positive integer $n$ we have $C_\varepsilon(Q_n f) \leq 2C_\varepsilon(f)$.

**Proof.** Since $Q_n f = f * s^n = s^n * f$, for any $\delta \leq \delta_0$ we have
\[
\int A(Q_n f, \delta, x) \, dx = \int \sup_{|x-y|<\delta} \left| \int_{-r_n}^{r_n} (f(x-t) - f(y-t)) s^n(t) \, dt \right| \, dx \\
\leq \int \int A(f, \delta, x-t) dx s^n(t) \, dt \leq \int A(f, \delta, x) \, dx.
\]

Condition (iii) on $\delta_0$ implies $\delta_0 < 1/6$, so we obtain
\[
A(f, \delta, x) = \begin{cases} 
A(f, \delta, -x) & \text{for } x \in [-1/4, 0), \\
A(f, \delta, x) & \text{for } x \in I, \\
A(f, \delta, 1-(x-1)) & \text{for } x \in (1, 5/4). 
\end{cases}
\]

Thus $C_\varepsilon(Q_n f) \leq 2C_\varepsilon(f)$.

Lemmas II.B.1, II.1, and I.B.2 imply the following

**Theorem II.B.** Let $\tau$ be as described above. If $2(\lambda^{-\varepsilon} + 4\lambda^{-1}) < 1$, then the analogous of Proposition I.B and Theorem I.B are true.

**III. Conjectures**

At the end of the paper we formulate two conjectures.

**Conjecture I. (P 1280)** If there exists a positive integer $k$ such that the mapping $\tau^k$ belongs to one of the classes discussed in the paper, then the analogues of all our propositions and theorems are true for the mapping $\tau$.

**Conjecture II. (P 1281)** Let for any positive integer $n$ the mapping $q^n: I \times I \to R^+$ be a measurable bounded function and let for all $x, y \in I$
\[
\int q^n(x, y) \, dy = \int q^n(x, y) \, dx = 1.
\]
Then all our results are valid for small stochastic perturbations generated by the family of densities \( q^n, \ n = 1, 2, \ldots \) (\( q^n \) discussed in the paper are examples of such densities).


**REFERENCES**


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