A PROBLEM OF INVARIANCE FOR LEBESGUE MEASURE

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Let \( q(\cdot, \cdot) \) be a metric on \( \mathbb{R}^n \) which is invariant under translations, i.e.,

\[ q(x+z, y+z) = q(x, y) \quad \text{for all } x, y, z \in \mathbb{R}^n, \]

and induces the usual topology of \( \mathbb{R}^n \), i.e.,

\[ \lim_{\|x\| \to 0} q(0, x) = 0 \quad \text{and} \quad \lim_{\|x\| \to 0} \|x\| = 0, \]

where \( \|\cdot\| \) is the usual Euclidean norm in \( \mathbb{R}^n \).

For any sets \( A, B \subseteq \mathbb{R}^n \) we say that \( A \) is \( q \)-isometric to \( B \) if there exists a bijection \( f: A \to B \) such that \( q(f(x), f(y)) = q(x, y) \) for all \( x, y \in A \).

\( \lambda(\cdot) \) denotes the \( n \)-dimensional Lebesgue measure in \( \mathbb{R}^n \).

It was proved in [1] that

(*) For every two open sets \( A, B \subseteq \mathbb{R}^n \), if \( A \) is \( q \)-isometric to \( B \), then \( \lambda(A) = \lambda(B) \).

**Problem.** Is (*) true for all Borel sets \( A, B \subseteq \mathbb{R}^n \)? (P 1085)

We do not know the answer even for \( n = 1 \). We will prove that the answer is positive under additional conditions on \( q \).

**Theorem.** If there exist constants \( \alpha, \beta > 0 \) such that for every open \( q \)-ball \( B \) of diameter less than or equal to \( \beta \) there exists a parallelotope \( P \subseteq B \) such that \( \lambda(P) \geq \alpha \lambda(B) \), then \( q \)-isometric Borel sets have the same Lebesgue measure.

**Proof.** Let \( C \) be the unit cube in \( \mathbb{R}^n \). Let \( \mathcal{B} \) be the family of all open \( q \)-balls in \( \mathbb{R}^n \). For any \( t > 0 \), let \( E(t) \) be the least number of balls in \( \mathcal{B} \) of diameter \( t \) necessary to cover \( C \), and put \( \delta(t) = 1/E(t) \). Now, with coverings from \( \mathcal{B} \), we define the Hausdorff \( h \)-measure \( \mu_h \) on \( \mathbb{R}^n \) (see [3]).

Clearly, \( q \)-isometric Borel sets have the same \( \mu_h \)-measure. To prove our theorem it is enough to show that \( 0 < \mu_h(C) < \infty \). In fact, by the uniqueness of Haar measure, this implies \( \mu_h = c\lambda \) for some constant \( c > 0 \), and hence the \( q \)-invariance of \( \lambda \).
First, for any $t > 0$ there exist balls $B_1, \ldots, B_{E(t)} \in \mathcal{A}$ of $\mathfrak{g}$-diameter $t$ with $C \subseteq B_1 \cup \ldots \cup B_{E(t)}$. Hence

$$\mu_\mathfrak{g}(C) \leq \sum_{i=1}^{E(t)} h(t_i) = 1.$$ 

Next, let $B_1, \ldots, B_n \in \mathcal{A}$ cover $C$, and let the $\mathfrak{g}$-diameter of $B_i$ equal $t_i$. If $P$ is a parallelootope of sufficiently small diameter, then $C$ can be covered with less than $2/\lambda(P)$ translates of $P$. Hence, if $t_i$ is sufficiently small, then

$$E(t_i) < 2/\lambda(P_i) < 2/(a\lambda(B_i)),$$

where $P_i$ is the parallelootope in $B_i$ given by the assumption of the theorem. Thus

$$h(t_i) > a\lambda(B_i)/2.$$ 

Since $\lambda(B_i) \geq 1$, we have

$$\sum_{i=1}^{n} h(t_i) \geq a/2.$$ 

Therefore $\mu_\mathfrak{g}(C) \geq a/2$, which completes our proof.

**Corollary 1.** If there exist constants $\alpha, \beta > 0$ such that for every $\mathfrak{g}$-ball $B$ of diameter less than or equal to $\alpha$ there exist Euclidean balls $B_0$ and $B_1$ of Euclidean diameters $r_0$ and $r_1$, respectively, with

$$B_0 \subseteq B \subseteq B_1 \quad \text{and} \quad r_0/r_1 \geq \beta,$$

then $\mathfrak{g}$-isometric Borel sets have the same Lebesgue measure.

For the proof it is enough to check that the assumptions of the Theorem are satisfied.

The function $\mathcal{Q}(0, x)$ is continuous, but it could be very irregular at 0. We know only one positive property:

**Proposition.** There exist $\alpha, \beta > 0$ such that $\mathcal{Q}(0, x) \geq \beta \|x\|$ whenever $\|x\| < \alpha$.

**Proof.** Let $\varepsilon, \eta > 0$ be arbitrary and choose $\delta > 0$ such that $\|x\| < \delta$ implies $\mathcal{Q}(0, x) < \varepsilon$. Assuming the proposition false, we infer that there exists an $x$ with $0 < \|x\| < \delta$ and $\mathcal{Q}(0, x) < \eta \|x\|$. Then, by the triangle inequality,

$$\mathcal{Q}(0, [1/\|x\|]x) \leq [1/\|x\|] \mathcal{Q}(0, x) \leq [1/\|x\|] \eta \|x\| \leq \eta,$$

where $[\cdot]$ is the greatest integer function. Hence

$$\mathcal{Q}(0, x/\|x\|) \leq \mathcal{Q}(0, [1/\|x\|]x) + \varepsilon \leq \eta + \varepsilon.$$

It follows that there exist $y \in S^{n-1}$ (unit sphere in $\mathbb{R}^n$) with $\mathcal{Q}(0, y)$ as small as we wish. From compactness of $S^{n-1}$ and continuity of $\mathcal{Q}$ it follows that $\mathcal{Q}(0, y) = 0$ for some $y \in S^{n-1}$. But this is a contradiction.
COROLLARY 2. If there exist constants \( \gamma, \delta > 0 \) such that

\[
\varrho(0, x) \leq \gamma \|x\| \quad \text{for} \quad \|x\| < \delta
\]

or else there exists an \( a > 0 \) such that

\[
\varrho(0, x) < \varrho(0, y) \quad \text{for} \quad \|x\| < \|y\| < a,
\]

then \( \varrho \)-isometric Borel sets have the same Lebesgue measure.

Proof. Assume (1). Then, by the Proposition, there exist \( \sigma, \tau > 0 \) such that

\[
\sigma \|x\| \leq \varrho(0, x) \leq \gamma \|x\| \quad \text{for} \quad \|x\| < \tau.
\]

Thus Corollary 1 applies with \( a = \tau \) and \( \beta = \sigma/\gamma \).

Assume (2). Then open \( \varrho \)-balls of sufficiently small diameters are Euclidean, and Corollary 1 applies again. Thus the proof is completed.

Corollary 2 applies to such familiar metrizations of \( \mathbb{R}^n \) as metrizations by homogeneous norms or metrizations of the form

\[
\frac{\|x-y\|}{1+\|x-y\|} \quad \text{or} \quad \sqrt{\|x-y\|}.
\]

However, it is easy to produce metrizations to which Corollary 1 is applicable but Corollary 2 is not.

Other results related to this paper were proved in [2].

REFERENCES


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