ON THE EXTENSIONS
OF UNIFORMLY CONTINUOUS MAPPINGS

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In this note we consider the problem of extensions of uniformly continuous mappings in uniform spaces and in metric spaces. This problem has been investigated by several authors (see [3], [4], and [1]).

(E) Let $A$, $X$ and $Y$ be metric spaces (respectively, uniform spaces) such that $A$ is a closed subset of $X$ and let $f: A \to Y$ be a uniformly continuous mapping. Under what conditions can $f$ be extended to a uniformly continuous mapping $F$ from the whole space $X$ into $Y$?

In Section 1 we consider problem (E) in uniform spaces. Using some corollaries to Katětov's theorem we prove that if $Y$ is an injective locally convex space, then every bounded uniformly continuous mapping from a closed subset $A$ of a uniform space $X$ into $Y$ can be extended to the whole space $X$. Since $R^1$ is injective, this generalizes Katětov's theorem.

In Section 2 we consider problem (E) in metric spaces. It is shown that a metric space $Y \in \text{AEU} (\mathfrak{M})$ if and only if $Y \in \text{ARU} (\mathfrak{M})$ and diam$(Y) < \infty$. We note that, in the sense of Isbell [3] and [4], AEU-spaces are the same as ARU-spaces.

In [2] Borsuk proved that if $X$ is the union of two closed subsets $X_1$ and $X_2$ such that $X_1$, $X_2$ and $X_1 \cap X_2$ are AR(\mathfrak{M})-spaces, then so is $X$. An example in Section 3 shows that the corresponding proposition for ARU(\mathfrak{M})-spaces is generally false, but under some additional assumptions the proposition holds true.

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1. Some corollaries to Katětov's theorem. First we recall the following theorem of Katětov [5]:

1.1. Theorem. Let $R^1$ denote the real line and let $A$ be a subset of a uniform space $X$. Then every bounded uniformly continuous mapping $f$ from
A into \( \mathbb{R}^1 \) admits a bounded uniformly continuous extension \( F \) from the whole space \( X \) into \( \mathbb{R}^1 \). Moreover, the extension \( F \) of \( f \) satisfies the condition
\[
\sup_{x \in X} \{|F(x)|\} = \sup_{x \in A} \{|f(x)|\}.
\]

Now we prove some immediate consequences from Theorem 1.1 which will be used in the sequel.

Let \( D \) be any set. By \( m(D) \) we denote the space of all bounded real functions on \( D \) with the supremum norm. The following corollary is an immediate consequence of Katětov's theorem:

**1.2. Corollary.** For every uniform space \( X \) and for every bounded uniformly continuous mapping \( f \) from a subset \( A \) of \( X \) into \( m(D) \), there exists a uniformly continuous mapping \( F: X \to m(D) \) such that \( F|A = f \) and
\[
\sup_{x \in X} \{\|F(x)\|\} = \sup_{x \in A} \{\|f(x)\|\}.
\]

**1.3. Corollary (Isbell [3]).** Let \( A \) be a subset of a uniform space \( X \) and let \( \varrho \) be a bounded uniformly continuous pseudometric on \( A \). Then there exists a bounded uniformly continuous pseudometric \( \varrho^* \) on \( X \) such that \( \varrho^*(x, y) = \varrho(x, y) \) for every \( x, y \in A \) and
\[
\sup_{x, y \in X} \varrho^*(x, y) = \sup_{x, y \in A} \varrho(x, y).
\]

**Proof.** Let \( g: A \to m(A) \) be a mapping defined by \( (g(x))y = \varrho(x, y) \) for \( x, y \in A \). Then \( g \) is bounded uniformly continuous. Thus, by Corollary 1.2, there exists a uniformly continuous mapping \( G: X \to m(A) \) such that \( G|A = g \) and
\[
\sup_{x \in X} \{\|G(x)\|\} = \sup_{x \in A} \{\|g(x)\|\}.
\]

Setting \( \varrho^*(x, y) = \|G(x) - G(y)\| \), we get a uniformly continuous pseudometric on \( X \) having the required properties. This completes the proof.

**1.4. Remark.** Suppose that \( X \) is a metric space with a metric \( \delta \) and let \( \varrho \) be a bounded uniformly continuous pseudometric on a closed subset \( A \) of \( X \). Then there is a bounded uniformly continuous pseudometric \( \varrho^* \) on \( X \) extending \( \varrho \) such that \( A \) is also closed with respect to the pseudometric \( \varrho^* \).

Indeed, putting
\[
\tilde{\varrho}(x, y) = \max \{\varrho^*(x, y), |\delta(x, A) - \delta(y, A)|\},
\]
we get the uniformly continuous pseudometric \( \tilde{\varrho} \) having the required properties.

**1.5. Definition.** A locally convex space \( Y \) is said to be injective if whenever \( A \) and \( X \) are locally convex spaces such that \( A \) is a closed
subspace of \( X \) and \( T: A \to Y \) is a continuous linear operator, then there exists a continuous linear operator \( T': X \to Y \) which extends \( T \).

By the Hahn-Banach theorem, \( R^1 \) is injective. The following theorem is a generalization of Katětov’s theorem.

**1.6. Theorem.** Let \( A \) be a closed subset of a uniform space \( X \) and let \( Y \) be an injective locally convex space. Then every bounded uniformly continuous mapping \( f \) from \( A \) into \( Y \) can be extended to a uniformly continuous mapping \( \bar{F} \) from the whole space \( X \) into \( Y \).

**Proof.** Let \( \{d_a\}_{a \in I} \) be a family of pseudometrics inducing the uniformity of \( X \) and let \( \{p_\gamma\}_{\gamma \in J} \) be a family of pseudonorms inducing the topology of \( Y \).

For every \( \gamma \in J \), put
\[
U_\gamma = \{ y \in Y : p_\gamma(y) \leq 1 \}
\]
and let \( U^\circ_\gamma \) be the polar of \( U_\gamma \). Define a mapping \( H_\gamma: Y \to m(U^\circ_\gamma) \) by
\[
(H_\gamma y)\varphi = \varphi(y)
\]
for every \( y \in Y \) and \( \varphi \in U^\circ_\gamma \). It can easily be seen that \( H_\gamma \) is a continuous linear operator from \( Y \) into \( m(U^\circ_\gamma) \) such that
\[
\|H_\gamma(y)\| = p_\gamma(y)
\]
for every \( y \in Y \) and \( \gamma \in J \).

Let \( f: A \to Y \) be a bounded uniformly continuous mapping from \( A \) into \( X \) of a uniform space \( Y \). Then for every \( \gamma \in J \) the mapping \( g_\gamma = H_\gamma \circ f \) is also a bounded uniformly continuous mapping from \( A \) into \( m(U^\circ_\gamma) \). Thus, by Corollary 1.2, for every \( \gamma \in J \) there exists a bounded uniformly continuous mapping \( G_\gamma: X \to m(U^\circ_\gamma) \) such that \( G_\gamma | A = g_\gamma \) and
\[
\sup_{x \in X} \{\|G_\gamma(x)\|\} = \sup_{x \in A} \{\|g_\gamma(x)\|\}.
\]

Let \( L(A) \) and \( L(X) \) denote the linear spaces spanned formally by elements of \( A \) and \( X \), respectively. Then \( L(A) \subset L(X) \).

Given
\[
x = \sum_{i=1}^{n} \lambda_i x_i \in L(X), \quad x_i \in X \text{ and } \lambda_i \in R^1 \text{ for } 1 \leq i \leq n,
\]
put
\[
q_\gamma(x) = \max \left\{ \left\| \sum_{i=1}^{n} \lambda_i G_\gamma(x_i) \right\| , \sup_{\varphi \in \Phi} \left| \sum_{i=1}^{n} \lambda_i \varphi(x_i) \right| \right\},
\]
where
\[
\Phi = \{ \varphi \in C(X) : \varphi | A = 0, |\varphi(x) - \varphi(y)| \leq d_a(x, y) \}
\]
for some \( a \in I \) and for every \( x, y \in X \).
It is easy to see that, for every \( \gamma \in J \), \( q_{\gamma} \) is a pseudonorm on \( L(X) \). Putting
\[
M = \bigcap_{\gamma \in J} q_{\gamma}^{-1}(0),
\]
we easily see that \( M \subset L(A) \).

Finally, let \( E = L(X)/M \) and \( F = L(A)/M \). Then \( E \) and \( F \) are locally convex spaces, and \( F \) is a subspace of \( E \). Let us show that \( F \) is closed in \( E \).

In fact, let \( x \notin F \). Then
\[
x = \left[ \sum_{i=1}^{n} \lambda_i x_i \right],
\]
where \( x_i \in X \) for \( 1 \leq i \leq n \), and \( \lambda_1, \ldots, \lambda_n \) are different from zero. Since \( x \notin F \), we can assume that \( x_1 \notin A \). Let us set \( B = A \cup \{x_2, \ldots, x_n\} \). Then \( B \) is closed in \( X \). Thus there exists an \( a_0 \in I \) such that
\[
d_{a_0}(x_1, B) = \inf_{y \in B} \{d_{a_0}(x_1, y)\} > 0.
\]

Putting \( \varphi_0(x) = d_{a_0}(x, B) \) for every \( x \in X \), we easily see that \( \varphi_0 \in \Phi \) and \( \varphi_0(x_1) > 0 \).

For every \( y \),
\[
y = \left[ \sum_{i=1}^{k} \mu_i y_i \right] \in F, \quad y_1, \ldots, y_k \in A,
\]
we get, by (2),
\[
q_{\gamma}(x - y) = \left| \sum_{i=1}^{n} \lambda_i \varphi_0(x_i) - \sum_{i=1}^{k} \mu_i \varphi_0(y_i) \right| = |\lambda_1| \varphi_0(x_1) > 0,
\]
which shows that \( x \notin \overline{F} \). This proves that \( F \) is closed in \( E \).

Now we define a linear operator \( T : F \to Y \) by
\[
T \left( \left[ \sum_{i=1}^{n} \lambda_i x_i \right] \right) = \sum_{i=1}^{n} \lambda_i f(x_i).
\]

Then by (1) and (2) we have
\[
p_{\gamma}(T(z)) = p_{\gamma} \left( \sum_{i=1}^{n} \lambda_i f(x_i) \right) = \|H_{\gamma} \left( \sum_{i=1}^{n} \lambda_i f(x_i) \right)\|
\]
\[
= \left\| \sum_{i=1}^{n} \lambda_i g_{\gamma}(x_i) \right\| \leq q_{\gamma} \left( \sum_{i=1}^{n} \lambda_i x_i \right) = q_{\gamma}(z)
\]
for every \( z = \sum_{i=1}^{n} \lambda_i x_i \in F \).

Thus \( T \) is continuous.
Since \( Y \) is injective, there exists a continuous linear operator \( T' : E \to Y \) such that \( T'|F = T \).

Setting \( \widetilde{F}(x) = T'([x]) \) for every \( x \in X \), we get a uniformly continuous extension \( \widetilde{F} \) of \( f \). Thus the theorem is proved.

2. Spaces \( \text{AEU}(\mathfrak{M}) \) and \( \text{ARU}(\mathfrak{M}) \). The notions of \( \text{AEU} \) and \( \text{ARU} \) uniform spaces were introduced and investigated by Isbell [3] and [4]. In this section we consider metric spaces only, hence we use the following definitions which differ slightly from those of Isbell [3] and [4].

2.1. Definition. A metric space \( Y \) is called an \( \text{AEU}(\mathfrak{M}) \) if, whenever \( X \) is a metric space and \( A \) is a closed subset of \( X \), any uniformly continuous mapping from \( A \) into \( Y \) can be extended to a uniformly continuous mapping from the whole space \( X \) into \( Y \).

2.2. Definition. A metric space \( Y \) is said to be an \( \text{ARU}(\mathfrak{M}) \) if, whenever \( Y \) is a closed subset of a metric space \( X \), then there exists a uniformly continuous retraction \( R \) from \( X \) onto \( Y \).

By \( \bar{Y} \) we denote the completion of a given metric space \( Y \). We have the following

2.3. Proposition. If \( Y \) is an \( \text{AEU}(\mathfrak{M}) \) (respectively, an \( \text{ARU}(\mathfrak{M}) \) then so is \( \bar{Y} \).

Proof. Let \( Y \) be an \( \text{ARU}(\mathfrak{M}) \) and let \( Z \) be a metric space containing \( \bar{Y} \). Let us put

\[ A = \{a : Y \subset a \subset Z \text{ such that } Y \text{ is closed in } a\}. \]

For every \( A_1, A_2 \in A \) set \( A_1 \leq A_2 \) if and only if \( A_1 \subset A_2 \). Then \( A \) becomes a partially ordered set satisfying the conditions of the Kuratowski-Zorn lemma. Let \( X \) be a maximal element of \( A \). It is easy to see that \( X \) is dense in \( Z \). Since \( Y \) is an \( \text{ARU}(\mathfrak{M}) \), there is a uniformly continuous retraction \( R \) from \( X \) onto \( Y \). Since \( \bar{Y} \) is complete, the retraction \( R \) can uniquely be extended to a uniformly continuous retraction \( \tilde{R} \) from \( Z \) onto \( \bar{Y} \). So \( \bar{Y} \) is an \( \text{ARU}(\mathfrak{M}) \).

The same argument shows that \( \bar{Y} \) is an \( \text{AEU}(\mathfrak{M}) \) whenever \( Y \in \text{AEU}(\mathfrak{M}) \). This completes the proof.

Clearly, if \( Y \) is an \( \text{AEU}(\mathfrak{M}) \), then \( Y \) is an \( \text{ARU}(\mathfrak{M}) \).

2.4. Theorem. A metric space \( Y \) is an \( \text{AEU}(\mathfrak{M}) \) if and only if \( Y \) is an \( \text{ARU}(\mathfrak{M}) \) and \( \text{diam}(Y) < \infty \).

Proof. Let \( Y \) be an \( \text{AEU}(\mathfrak{M}) \). Then \( Y \) is an \( \text{ARU}(\mathfrak{M}) \) and we have to show that \( \text{diam}(Y) < \infty \).

Assume, on the contrary, that \( \text{diam}(Y) = \infty \). Let \( \{x_n\} \) be a sequence of points in \( Y \) such that

\[ d(x_n, x_{n+1}) \geq n. \]
Let \( X = \mathbb{R}^1 \) (\( \mathbb{R}^1 \) being the real line) and let \( A = \mathbb{N} \) (\( \mathbb{N} \) being the set of all natural numbers). Define a mapping \( f: \mathbb{N} \to Y \) by \( f(n) = x_n \). Then \( f \) is uniformly continuous on \( \mathbb{N} \) and \( \mathbb{N} \) is closed in \( \mathbb{R}^1 \). Let \( F: \mathbb{R}^1 \to Y \) be a uniformly continuous extension of \( f \). By a lemma of Lindenstrauss [7] there exists an \( L > 0 \) such that
\[
\delta(f(x), f(y)) \leq L|x-y|
\]
for every \( x, y \in \mathbb{R}^1 \) with \( |x-y| \geq 1 \). Then we get
\[
\delta(f(n+1), f(n)) \leq L
\]
for every \( n \in \mathbb{N} \), a contradiction with (4). This shows that \( \text{diam}(Y) < \infty \).

Conversely, assume that \( Y \) is an ARU(\( \mathbb{R} \)) and \( \text{diam}(Y) < \infty \). Let \( f \) be a uniformly continuous mapping from a closed subset \( A \) of a metric space \( X \) into \( Y \). First we consider a special case where \( f \) is an isometric embedding.

Let \( Q = X \cup Y \) and let \( Z = Q/\sim \), where \( \sim \) is the equivalence relation on \( Q \) defined by \( x \sim y \) if and only if \( y = f(x) \) or \( y = x \). Setting
\[
\varrho(x, y) = \begin{cases} \delta_X(x, y) & \text{if } x, y \in X, \\ \delta_Y(x, y) & \text{if } x, y \in Y, \\ \inf_{t \in A} \{ \delta_X(x, t) + \delta_Y(y, f(t)) \} & \text{if } x \in X \text{ and } y \in Y, \end{cases}
\]
where \( \delta_X \) and \( \delta_Y \) denote metrics on \( X \) and \( Y \), respectively, we easily see that \( \varrho \) is a metric on \( Z \), and \( Y \) is closed in \( Z \). Let \( R \) be a uniformly continuous retraction from \( Z \) onto \( Y \). Then \( F = R \circ i \), where \( i: X \to Z \) is the natural inclusion, is a uniformly continuous extension of \( f \).

Now, let \( f: A \to Y \) be an arbitrary uniformly continuous mapping. Putting
\[
\check{h}(x, y) = \delta_Y(f(x), f(y)),
\]
we get a bounded uniformly continuous pseudometric on \( A \). By Remark 1.4, there exists a uniformly continuous pseudometric \( \check{h} \) on \( X \) such that \( \check{h}|A \times A = \check{h} \) and \( A \) is closed with respect to the pseudometric \( \check{h} \) of \( X \).

Let \( E = X/\check{h} \) and \( B = A/\check{h} \subseteq E \). Then \( E \) is a metric space with the metric \( \check{h} \) induced by \( \check{h} \), and \( B \) is a closed subset in \( E \). It is easy to see that the mapping \( g: B \to Y \) induced by \( f \) is an isometric embedding. Thus, using the proof above, we get a uniformly continuous mapping \( G \) from \( E \) into \( Y \) such that \( G|B = g \). Setting \( F = G \circ k \), where \( k: X \to E \) is the quotient mapping, we easily see that \( F \) is uniformly continuous with respect to the metric \( \delta_X \) and \( F(x) = f(x) \) for every \( x \in A \). Thus the theorem is proved.
2.5. Remark. Let \((X, d)\) be a metric space. For every \(\varepsilon > 0, x \in X\) and \(n \in \mathbb{N}\), put
\[
B_1(x, \varepsilon) = \{y \in X : d(x, y) \leq \varepsilon\}
\]
and define \(B_n(x, \varepsilon)\) by induction:
\[
B_n(x, \varepsilon) = \{y \in X : d(y, z) \leq \varepsilon \text{ for some } z \in B_{n-1}(x, \varepsilon)\}.
\]
A metric space \((X, d)\) is said to be uniformly bounded if there is a point \(x_0 \in X\) such that for every \(\varepsilon > 0\) there exists an \(n \in \mathbb{N}\) such that \(B_n(x_0, \varepsilon) = X\).

A similar argument as in the proof of Theorem 2.4 shows that every \(\text{AEU}(\mathcal{M})\)-space is uniformly bounded.

2.6. Remark. It is known (see, e.g., [1]) that every Lipschitz mapping \(f\) from a subset \(A\) of a metric space \(X\) into \(\mathbb{R}^1\) can be extended to a Lipschitz mapping \(F\) from \(X\) into \(\mathbb{R}^1\). In particular, we infer that \(\mathbb{R}^1\) is an \(\text{ARU}(\mathcal{M})\); however, it is not an \(\text{AEU}(\mathcal{M})\).

2.7. Remark. Let us put \(\rho(x, y) = \min\{1, |x - y|\}\) for every \(x, y \in \mathbb{R}^1\). Then from Theorem 2.4 we infer that \((\mathbb{R}^1, \rho)\) is not an \(\text{ARU}(\mathcal{M})\); however, \((\mathbb{R}^1, \rho)\) and \((\mathbb{R}^1, |\cdot|)\) are uniformly equivalent.

2.8. Remark. Isbell [4] showed that, for \(-\infty < a < b < \infty\), \((a, b)\) is an \(\text{ARU}(\mathcal{M})\). Thus from Theorem 2.4 we see that \((a, b)\), [\(a, b]\), \((a, b)\) and \([a, b]\) are \(\text{AEU}(\mathcal{M})\)-spaces.

3. The union of two \(\text{AEU}(\mathcal{M})\)-spaces.

3.1. Theorem. Let \((X, \rho)\) be a metric space and let \(X_0, X_1, X_2\) be closed subsets of \(X\) such that \(X = X_1 \cup X_2\) and \(X_0 = X_1 \cap X_2 \neq \emptyset\). Assume that \(X_1, X_0, X_2 \in \text{AEU}(\mathcal{M})\). Then \((X, \rho) \in \text{AEU}(\mathcal{M})\) if and only if the metric \(\bar{d}\) on \(X\) defined by
\[
\bar{d}(x, y) = \begin{cases} 
\rho(x, y) & \text{if } (x, y) \in X_i \times X_i \text{ for } i = 1, 2, \\
\inf_{t \in X_0} \{\rho(x, t) + \rho(y, t)\} & \text{otherwise}
\end{cases}
\]
is uniformly equivalent to \(\rho\).

Proof. First we assume that \(X_0, X_1, X_2 \in \text{AEU}(\mathcal{M})\). Then, by Theorem 2.4, \(\operatorname{diam} \rho(X) < \infty\). By the definition of \(\bar{d}\), \(\operatorname{diam}_{\bar{d}}(X) < \infty\). In order to prove that \((X, \rho) \in \text{AEU}(\mathcal{M})\) it now suffices to show that \((X, d)\) is an \(\text{ARU}(\mathcal{M})\).

Let \(Z\) be a metric space containing \((X, d)\) isometrically as a closed subset. By a theorem of Kuratowski and Wojdysławski (see, e.g., [2],
we may assume without loss of generality that \( Z \) is a convex set lying in a normed space. Let us put

\[
Z_0 = \{ z \in Z : d(z, X_1) = d(z, X_2) \},
\]

\[
Z_1 = \{ z \in Z : d(z, X_1) < d(z, X_2) \},
\]

\[
Z_2 = \{ z \in Z : d(z, X_1) > d(z, X_2) \}.
\]

Clearly, \( Z = Z_0 \cup Z_1 \cup Z_2 \) and \( X_i \cap Z_0 = X_0 \) for \( i = 1, 2 \). Since \( X_0 \) is closed in \( Z_0 \), there exists a uniformly continuous retraction \( R_0 \) from \( Z_0 \) onto \( X_0 \). Let

\[
R_i : X_i \cup Z_0 \to X_i \cup X_0 = X_i
\]

be a mapping defined by

\[
R_i(z) = \begin{cases} 
  z & \text{for } z \in X_i, \\
  R_0(z) & \text{for } z \in Z_0.
\end{cases}
\]

Observe that \( R_i \) is uniformly continuous for \( i = 1, 2 \).

Indeed, to show this it is enough to prove that if \( x \in X_i \) and \( y \in Z_0 \) are sufficiently close, then \( R_1(x) = x \) is closed to \( R_1(y) = R_0(y) \).

Given any \( \varepsilon > 0 \), let \( \delta \in (0, \varepsilon/6) \) be such that if \( x, y \in Z_0 \) and \( d(x, y) < 4\delta \), then \( d(R_0(x), R_0(y)) < \varepsilon/2 \). Let \( x \in X_1 \) and \( y \in Z_0 \) be such that \( d(x, y) < \delta \). We shall show that \( d(R_1(x), R_1(y)) < \varepsilon \). By the definition of \( Z_i \) there exists a \( z \in X_2 \) such that \( d(z, y) < \delta \). By the definition of \( d \) there is a \( t \in X_0 \) such that

\[
d(x, t) + d(z, t) \leq d(x, z) + \delta \leq d(x, y) + d(y, z) + \delta < 3\delta.
\]

Thus we have

\[
d(x, t) < 3\delta \quad \text{and} \quad d(z, t) < 3\delta.
\]

Therefore

\[
d(t, y) \leq d(t, x) + d(x, y) < 3\delta + \delta = 4\delta.
\]

Hence

\[
d(R_0(t), R_0(y)) < \frac{\varepsilon}{2}.
\]

Consequently,

\[
d(R_1(x), R_1(y)) = d(x, R_0(y)) \leq d(x, t) + d(t, R_0(y))
\]

\[
= d(x, t) + d(R_0(t), R_0(y)) < 3 \frac{\varepsilon}{6} + \frac{\varepsilon}{2} = \varepsilon.
\]

The uniform continuity of \( R_i \) is established.

Since \( X_i \in \Delta EU(\mathcal{M}) \) and \( X_i \cup Z_0 \) is closed in \( Z_i \cup Z_0 \), we infer that the mapping \( R_i \) can be extended to a uniformly continuous mapping \( f_i \) from \( Z_i \cup Z_0 \) into \( X_i \).
Now define a retraction \( R \) from \( Z \) onto \( X \) by

\[
R(z) = f_i(z) \quad \text{if } z \in Z_i \cup Z_0 \text{ for } i = 1, 2.
\]

Since \( f_i \) is uniformly continuous for every \( i = 1, 2 \), to show that \( R \) is uniformly continuous it is enough to prove that if \( x \in Z_1 \cup Z_0 \) and \( y \in Z_2 \cup Z_0 \) are sufficiently close, then \( R(x) \) is closed to \( R(y) \).

Indeed, let \( \varepsilon > 0 \) be given. Since the function \( f_i \) is uniformly continuous, there exists a \( \delta > 0 \) such that if \( x, y \in Z_1 \cup Z_0 \) and \( d(x, y) < \delta \), then \( d(f_i(x), f_i(y)) < \varepsilon/2 \) for every \( i = 1, 2 \).

Let \( x \in Z_1 \cup Z_0 \) and \( y \in Z_2 \cup Z_0 \) be such that \( d(x, y) < \delta \). It follows from the definition of \( Z_i \) that there is an \( \alpha \in [0, 1] \) such that

\[
z = \alpha x + (1 - \alpha) y \in Z_0.
\]

Since \( d(x, z) \leq d(x, y) < \delta \) and \( d(y, z) \leq d(x, y) < \delta \), we infer that

\[
d(R(x), R(y)) \leq d(R(x), R(z)) + d(R(z), R(y))
\]

\[
= d(f_1(x), f_1(z)) + d(f_2(x), f_2(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Thus \( R \) is a uniformly continuous retraction.

Conversely, assume that \( (X, \varrho) \in AEU(\mathbb{R}) \). Let us prove that \( \varrho \) and \( d \) are uniformly equivalent.

Again, we can assume that \( (X, \varrho) \) is a closed subset of a convex set \( Z \) lying in a normed space. Let \( R: Z \to X \) be a uniformly continuous retraction. To show that \( d \) and \( \varrho \) are uniformly equivalent it suffices to prove that if \( \{x_n\} \subset X_1 \) and \( \{y_n\} \subset X_2 \) are such that \( \varrho(x_n, y_n) \to 0 \), then \( d(x_n, y_n) \to 0 \).

In fact, let

\[
[x_n, y_n] = \{z \in Z: z = tx_n + (1 - t)y_n, 0 \leq t \leq 1\}.
\]

We easily see that, for every \( n \in N \), there exists a \( z_n \in [x_n, y_n] \) such that \( R(z_n) \in Z_0 \). Since

\[
\varrho(x_n, R(z_n)) \leq \text{diam}(R[x_n, y_n]) \quad \text{and} \quad \varrho(y_n, R(z_n)) \leq \text{diam}(R[x_n, y_n]),
\]

we infer that

\[
d(x_n, y_n) \leq \varrho(x_n, R(z_n)) + \varrho(R(z_n), y_n) \leq 2 \text{diam}(R[x_n, y_n]) \to 0.
\]

Thus the theorem is proved.

3.2. Example. Let \( ABC \) be a triangle in the plane \( \mathbb{R}^2 \) and let

\[
X = [AB] \cup (BC] \cup [CA], \quad X_1 = [AC] \cup [CB], \quad X_2 = [CA] \cup [AB].
\]
Then $X_1, X_2$ and $X_1 \cap X_2 \in \text{AEU}(\mathcal{M})$ (see Remark 2.8). Moreover, $X_1$ and $X_2$ are closed in $X$, but the completion of $X$ is not an $\text{AR}(\mathcal{M})$ (in the sense of Borsuk [2]). Therefore, by Proposition 2.3, $X$ is not an $\text{ARU}(\mathcal{M})$.

3.3. COROLLARY. In the notation of Theorem 3.1, if at least one of the subsets $X_1, X_2$ is compact, then $(X, \varrho) \in \text{AEU}(\mathcal{M})$.

Indeed, it is easy to see that if one of the subsets $X_1, X_2$ is compact, then the metric $d$ is uniformly equivalent to $\varrho$.

REFERENCES


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