MOST MARKOV OPERATORS ON $C(X)$
ARE QUASI-COMPACT AND UNIQUELY ERGODIC

BY

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1. Uniquely ergodic Markov operators. We denote by $C(X)$ the Banach space of continuous real-valued functions on a compact Hausdorff space $X$. A linear operator $T$ on $C(X)$ which is positive ($f \geq 0 \Rightarrow Tf \geq 0$) and takes 1 into 1 is said to be Markov. A probability (Radon) measure $\mu$ on $X$ is said to be $T$-invariant if $T^\mu \mu = \mu$. By the Markov–Kakutani fixed point theorem the set of all invariant probability measures is non-empty. $T$ is uniquely ergodic if there exists only one invariant probability measure. It is well known (see, e.g., [8]) that $T$ is uniquely ergodic if and only if the Cesàro means

$$A_n = n^{-1}(I + T + \ldots + T^{n-1})$$

converge in the strong operator topology to a one-dimensional projection which necessarily is of the form $E_\mu f = (\mu, f) 1$, where $\mu$ is a probability measure on $X$.

$T$ is said to be uniformly ergodic if the norm closed convex hull of \{ $T^n; n \geq 0$ \} contains an operator $P$ such that $TP = P$ (see [7]). If $T$ is a uniformly ergodic Markov operator on $C(X)$, then by, e.g., Proposition 1 in [7], the Cesàro means $A_n$ converge in the norm topology.

A Markov operator $T$ is said to be quasi-compact if $\|T^n - K\| < 1$ for some positive integer $n$ and some compact operator $K$. It is well known (see [9]) that quasi-compactness implies uniform ergodicity.

We shall use the above results to prove Lemma 2.

LEMMA 1. Let a Markov operator $T$, a probability measure $\mu$, and $0 \leq \alpha < 1$ be given. Then the iterates of the operator $S = \alpha T + (1-\alpha)E_\mu$ converge in the norm topology to a one-dimensional projection.

Proof. It is easily seen that

$$S^n = \alpha^n T^n + E_{\mu^n} \quad \text{for some } \mu_n \in C^*(X).$$

Let $\nu$ be an $S$-invariant probability measure. Then

$$\|S^n - E_\nu\| \leq \alpha^n + \|E_\nu - E_{\mu^n}\| = \alpha^n + \|E_\nu S^n - E_{\mu^n}\|$$

$$= \alpha^n + \|\alpha^n E_\nu T^n\| \leq 2\alpha^n.$$
By Lemma 1, the uniquely ergodic quasi-compact Markov operators are norm dense in the set of Markov operators on \( C(X) \).

**Lemma 2.** If there exists a sequence \( (\mu_k) \) of probability measures on \( X \) such that

\[
\| A_{n_k} - E_{\mu_k} \| \rightarrow 0,
\]

then \( T \) is quasi-compact and uniquely ergodic.

**Proof.** For sufficiently large \( k \) the Markov operators \( A_{n_k} \) are quasi-compact. Hence, for those \( k \), the Cesàro means \( m^{-1} \sum_{j=0}^{m-1} A_{n_k}^j \) converge in the norm operator topology to a projection \( P_k \). Clearly, \( P_k A_{n_k} = P_k \). Moreover, we have \( P_k A_n = A_n P_k \) for all \( k \) and \( n \). Hence

\[
\| P_k - E_{\mu_k} \| = \| P_k A_{n_k} - P_k E_{\mu_k} \| \leq \| A_{n_k} - E_{\mu_k} \| \rightarrow 0.
\]

Since

\[
\| P_j - P_k A_{n_j} \| \leq \| P_j - E_{\mu_j} \| + \| P_k E_{\mu_j} - P_k A_{n_j} \| \rightarrow 0
\]

uniformly in \( k \), and

\[
\| P_k A_{n_j} - P_k \| \leq \| P_k - E_{\mu_k} \| + \| E_{\mu_k} - P_k A_{n_j} \|
\]

\[
= \| P_k - E_{\mu_k} \| + \| A_{n_j} E_{\mu_k} - A_{n_j} P_k \| \rightarrow 0
\]

uniformly in \( j \), the sequence \( (P_k) \) is Cauchy. As \( A_{n_k}^j \) belong to the convex hull \( \text{co} \{ T^n; \ n \geq 0 \} \) for all \( k, j \), the operators \( P_k \) and \( P := \lim P_k = \lim E_{\mu_k} \) belong to the closure of \( \text{co} \{ T^n; \ n \geq 0 \} \). Furthermore, \( PT = TP = P \), so \( T \) is uniformly ergodic. Hence \( T \) is quasi-compact and uniquely ergodic since \( P = \lim E_{\mu_k} = \lim A_k \) is one-dimensional (see, e.g., Proposition 1 in [7] and Theorem 1 in [6]).

**Theorem 1.** The uniquely ergodic quasi-compact operators form a dense \( G_\delta \)-set for the norm topology in the set of Markov operators on \( C(X) \).

**Proof.** Let \( \mathcal{U} \) be the set of all Markov operators \( T \) for which the Cesàro means \( A_n \) converge in the norm topology to a one-dimensional projection. By Theorem 1 in [6], \( \mathcal{U} \) consists of all uniquely ergodic quasi-compact operators. Let

\[
\mathcal{U}_1 = \bigcap_{k} \bigcap_{n} \bigcup_{m \geq n} \bigcup_{\mu} \{ T; \ T \text{ is Markov and } \| A_m - E_{\mu} \| < 1/k \},
\]

where \( \bigcup \) is the union over all probability measures on \( X \). Clearly, \( \mathcal{U} \subset \mathcal{U}_1 \) and \( \mathcal{U}_1 \) is a \( G_\delta \)-set. By Lemma 2, each \( T \) from \( \mathcal{U}_1 \) is uniquely ergodic and quasi-compact, so \( \mathcal{U} = \mathcal{U}_1 \). By the remark following Lemma 1, \( \mathcal{U} \) forms a norm dense subset.
2. Stochastic operators on $L^1$. In this section we apply Theorem 1 to some stochastic operators. Let $(\Omega, \Sigma, m)$ be a probability space. A linear operator $T$ on $L^1(m)$ is said to be {f stochastic} if $T$ is positive (i.e., $f \geq 0 \Rightarrow Tf \geq 0$) and satisfies the equality $T^*1 = 1$ (or, equivalently, $f \geq 0 \Rightarrow \|Tf\| = \|f\|$). The set of all stochastic operators will be denoted by $\mathcal{S}$.

A stochastic operator $T$ is said to be {f conservative} if $\sum_{n=1}^\infty T^n f = \infty$ a.e. for every (or, equivalently, for some) strictly positive function $f \in L^1(m)$ (see, e.g., [2], Chapter 2). The set of all conservative operators in $\mathcal{S}$ will be denoted by $\mathcal{C}$. The set of all stochastic operators $T$ for which $T^*1_A = 1_A$ implies $m(A)(1 - m(A)) = 0$ ($A \in \Sigma$) will be denoted by $\mathcal{O}$.

In this section we discuss operators from $\mathcal{S} \cap \mathcal{C}$, the set of conservative and ergodic stochastic operators. In [5] and [1] topological properties of $\mathcal{C} \cap \mathcal{O}$ have been studied. It has been shown (for $\Omega = [0, 1]$) that conservative and ergodic operators form a dense $G_\delta$-set for both strong operator and norm topologies in $\mathcal{S}$. The Harris operators form an important subset of $\mathcal{C} \cap \mathcal{O}$. Let $T \in \mathcal{C} \cap \mathcal{O}$ and $T^n = Q_n + R_n$, where $Q_n$ is a positive integral operator with kernel $q_n(x, y)$, and $R_n$ is such that there is no non-zero integral operator $K$ with $0 \leq K \leq R_n$ (see [2], Chapter 5, and [3]). $T$ is said to be a {f Harris operator} if $Q_n \neq 0$ for some $n \geq 1$ (see [2] and [3]).

By the Gelfand–Naimark theorem there is a 0-dimensional compact Hausdorff space $\hat{X}$ such that $L_\infty(\hat{X})$ and $C(\hat{X})$ are isometrically isomorphic. For each $T \in \mathcal{S}$ let $\hat{T}$ denote the corresponding operator on $C(\hat{X})$. Clearly, $\hat{T}$ is a Markov operator. If $T \in \mathcal{C}$, then $\hat{T}$ is uniquely ergodic and quasi-compact if and only if $T \in \mathcal{C} \cap \mathcal{O}$ and $T^*$ is quasi-compact on $L_\infty(m)$ (see Theorem 4.1 in [4]).

Now we can prove the following

**Theorem 2.** The conservative and ergodic quasi-compact operators form a dense $G_\delta$-set for the norm topology in $\mathcal{S}$.

**Proof.** From Theorem 1 and the remark above, the conservative and ergodic quasi-compact operators form a $G_\delta$-subset of $\mathcal{C}$ for the norm topology. As in [5] (Lemma 2) we can show that the conservative operators form a $G_\delta$-set for the norm (even for strong operator) topology in $\mathcal{S}$. Hence the conservative and quasi-compact operators, being an intersection of two $G_\delta$'s, form a $G_\delta$-set for the norm topology in $\mathcal{S}$.

To prove the proof, it remains to show that this set is norm dense in $\mathcal{S}$. Let $T \in \mathcal{S}$. Then for each $\alpha$ ($0 \leq \alpha < 1$) the operator $S = \alpha T + (1 - \alpha) E_m$ is conservative, ergodic and quasi-compact. Indeed, by Lemma 2 in [1], $S$ has an equivalent invariant probability measure $\mu$, and therefore $S$ is conservative. Clearly, $S$ is ergodic. Similarly as in the proof of Lemma 1, $S^n$ converge to $E_\mu$ in the norm topology, so $S$ is quasi-compact. Clearly, $S = S(\alpha) \to T$ in the norm topology.
Now, from Theorem 4.1 in [4] we obtain

**Corollary.** The set of Harris operators is norm residual in the set of stochastic operators.

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**References**


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