ON ADMISSIBLE WHITNEY MAPS

BY

HISAO KATO (HIROSHIMA)

1. Introduction. Throughout this paper, the word compactum means a compact metric space. A continuum is a connected compactum. Let \( X \) be a metric space with metric \( \rho \). The hyperspaces of \( X \) are the spaces

\[
2^X = \{ A \mid A \text{ is a nonempty and compact subset of } X \}
\]

and

\[
C(X) = \{ A \in 2^X \mid A \text{ is connected} \}
\]

which are metrized with the Hausdorff metric \( \rho_H \), i.e.,

\[
\rho_H(A, B) = \max_{a \in A} \{ \sup_{b \in B} \rho(a, b) \}
\]

A Whitney map for a hyperspace \( \mathcal{H} = 2^X \) or \( C(X) \) is a continuous function \( \omega : \mathcal{H} \to [0, \omega(X)] \) such that \( \omega(\{x\}) = 0 \) for each \( x \in X \), and if \( A, B \in \mathcal{H} \), \( A \subset B \) and \( A \neq B \), then \( \omega(A) < \omega(B) \). In [12], Whitney showed that for any metric space \( (X, \rho) \) there always exists a Whitney map for \( \mathcal{H} = 2^X \) or \( C(X) \). We recall the construction. Let \( A \in \mathcal{H} \). For each \( n \geq 2 \) let

\[
F_n(A) = \{ K \subset A \mid K \neq \emptyset \text{ and the cardinality of } K \text{ is } \leq n \}.
\]

Also, define \( \lambda_n : F_n(A) \to [0, \infty) \) by letting

\[
\lambda_n(\{a_1, a_2, \ldots, a_n\}) = \min \{ \rho(a_i, a_j) \mid i \neq j \}
\]

for all \( \{a_1, a_2, \ldots, a_n\} \in F_n(A) \), and let

\[
\omega_n(A) = \sup \lambda_n(F_n(A)).
\]

Then

\[
(\ast) \quad \omega(A) = \sum_{n=2}^{\infty} \omega_n(A)/2^{n-1}.
\]

The notion of Whitney map is a convenient and important tool in order to study hyperspaces theory. It is of interest to obtain information about Whitney levels \( \omega^{-1}(t) \) \((0 < t < \omega(X)) \) and to determine those properties
which are preserved by the convergence of positive Whitney levels $\omega^{-1}(t_a)$ ($t_a > 0$) to the zero level $\omega^{-1}(0) = X$. In [3] and [11], Curtis, Schori and West proved that for any Peano continuum (= locally connected continuum) $X$, $2^X$ is homeomorphic to the Hilbert cube $Q = [-1, +1]^\infty$, and if $X$ contains no free arc, $C(X)$ is homeomorphic to $Q$. In [5], Goodykoontz and Nadler introduced the notion “admissible Whitney map”. A Whitney map $\omega$ for $\mathcal{H} = 2^X$ or $C(X)$ is admissible [5] if there is a homotopy $h: \mathcal{H} \times I \to \mathcal{H}$ satisfying the following conditions:

(A1) for all $A \in \mathcal{H}$,

$$h(A, 1) = A, \quad h(A, 0) \in F_1(X) = \{\{x\} | x \in X\};$$

(A2) if $\omega(h(A, t)) > 0$ for some $A \in \mathcal{H}$, $t \in I$, then

$$\omega(h(A, s)) < \omega(h(A, t)) \quad \text{whenever} \quad 0 \leq s < t \leq 1.$$  

In [5], it was shown that if $X$ is either a compact starshaped subset of a Banach space or a 1-dimensional AR (= dendrite), then there exist admissible Whitney maps for $\mathcal{H} = 2^X$ and $C(X)$. By using the notion of admissible Whitney map, Goodykoontz and Nadler [5] proved the following

(1.1) Let $X$ be a Peano continuum and let $\omega$ be an admissible Whitney map for $\mathcal{H} = 2^X$ or $C(X)$. If $\mathcal{H} = C(X)$, assume that $X$ contains no free arc. Then, for any $t \in (0, \omega(X))$, $\omega^{-1}(t)$ is a Hilbert cube.

A Whitney map $\omega$ for $\mathcal{H} = 2^X$ or $C(X)$ is strongly admissible [6] if there is a homotopy $h: \mathcal{H} \times I \to \mathcal{H}$ satisfying (A1), (A2) and

(A3) $h(\{x\}, t) = \{x\}$ for each $x \in X$ and $t \in I$.

In [6], we proved the following

(1.2) Let $X$ be a Peano continuum and let $\omega$ be an admissible Whitney map for $\mathcal{H} = 2^X$ or $C(X)$. If $\mathcal{H} = C(X)$, assume that $X$ contains no free arc. Then the restriction

$$\omega|\omega^{-1}((0, \omega(X))): \omega^{-1}((0, \omega(X))) \to (0, \omega(X))$$

of $\omega$ to $\omega^{-1}((0, \omega(X)))$ is a trivial bundle map with Hilbert cube fibers. Moreover, if $X$ is the Hilbert cube $Q$, then there is a Whitney map $\omega$ for $\mathcal{H} = 2^Q$ or $C(Q)$ such that $\omega|\omega^{-1}((0, \omega(Q)))$ is a trivial bundle map with Hilbert cube fibers. Also, if $X$ is the $n$-sphere $S^n$ ($n \geq 1$), then there is a Whitney map $\omega$ for $\mathcal{H} = 2^{S^n}$ ($n \geq 1$) or $C(S^n)$ ($n \geq 2$) such that, for some $t_0 \in (0, \omega(S^n))$, $\omega|\omega^{-1}((0, t_0))$ is a trivial bundle map with $S^n \times Q$ fibers.

The purpose of this paper is to prove the following:

(1) Let $P$ be a finite collapsible polyhedron and let $\mathcal{H} = 2^P$ or $C(P)$. If $\mathcal{H} = C(P)$, assume that $P$ contains no free arc. Then there is a Whitney map $\omega$ for $\mathcal{H}$ such that $\omega|\omega^{-1}((0, \omega(P)))$ is a trivial bundle map with Hilbert cube fibers.
Let $K$ be a cubical complex and let $P = |K|$. Let $\mathcal{H} = 2^P$ or $C(P)$. If $\mathcal{H} = C(P)$, assume that $P$ contains no free arc. If $K$ is locally regular collapsible, then there is a Whitney map $\omega$ for $\mathcal{H}$ such that, for some $t_0 \in (0, \omega(P))$, $\omega \omega^{-1}((0, t_0))$ is a trivial bundle map with $P \times Q$ fibers.

2. Whitney maps and hyperconvex metric spaces. A metric space $(X, \rho)$ is hyperconvex (or injective) [1] if $\rho$ is convex and any collection of solid spheres in pairwise intersection in $X$ has a common point. In [1], Theorem 3, it was proved that a metric space $(X, \rho)$ is hyperconvex if and only if every mapping which increases no distance from a subset of any metric space $Y$ to $X$ can be extended, increasing no distance, over $Y$.

First, we show the following

**Theorem.** Let $(X, \rho)$ be a hyperconvex metric compactum. Suppose that $\omega$ is the Whitney map for $\mathcal{H} = 2^P$ or $C(X)$ which is defined by $(\ast)$ and the metric $\rho$. Then $\omega$ is strongly admissible.

**Proof.** It is well known that there exists a Banach space $B$ with norm $\| \cdot \|$ such that $B$ contains $X$ and, for any $x, y \in X$, $\rho(x, y) = \|x - y\|$ (see [8]). Let $\omega'$ be the Whitney map for $\mathcal{H}' = 2^B$ or $C(B)$ as is defined by $(\ast)$ and the metric $\rho'$, where $\rho'(b, b') = \|b - b'\|$ for $b, b' \in B$. Since $X$ is an AR (see [1]), there is a retraction $r: \mathcal{H} \to X$, i.e., $r(\{x\}) = x$ for each $x \in X$. Let $A \in \mathcal{H}$. Define a homotopy $h_A: A \times I \to B$ by

$$h_A(a, t) = (1 - t) \cdot r(A) + t \cdot a \quad \text{for each } a \in A, t \in I.$$ 

Also, define a homotopy $h': \mathcal{H} \times I \to \mathcal{H}'$ by

$$h'(A, t) = \{h_A(a, t) \mid a \in A\} \quad \text{for each } A \in \mathcal{H}, t \in I.$$ 

Since $(X, \rho)$ is hyperconvex, there is a contraction $f: B \to X$, i.e.,

$$\rho(f(y), f(z)) \leq \rho'(y, z) = \|y - z\| \quad \text{for } y, z \in B.$$ 

If $x, y \in A$, $x \neq y$, and $0 \leq t' < t \leq 1$, then

$$\rho'(h_A(x, t'), h_A(y, t')) = \|t' - (x - y)\| < \|t(x - y)\|$$

$$= \rho'(h_A(x, t), h_A(y, t)).$$ 

Hence, if $A$ is nondegenerate, by (3) we have

$$\omega'(h'(A, t')) < \omega'(h'(A, t)) \quad \text{for } 0 \leq t' < t \leq 1$$

(see [5], (2.13)). Since $f$ is a contraction, we can easily see

$$\omega\left(f\left(h'(A, t)\right)\right) = \omega\left(f\left(h'(A, t)\right)\right) \leq \omega'(h'(A, t)) \quad \text{for each } A \in \mathcal{H}, t \in I.$$ 

Consider the function $K_\rho: \mathcal{H} \times [0, \infty) \to \mathcal{H}$ defined by

$$K_\rho(A, s) = \{x \in X \mid \rho(A, x) \leq s\} \quad \text{for each } A \in \mathcal{H}, s \in [0, \infty).$$
Since \( q \) is convex, \( K_q \) is continuous (see [5], (1.2)). For each \( A \in \mathcal{H} \) and \( t \in I \), there is the minimal number \( m(A, t) \geq 0 \) such that
\[
\omega\left( K_q\left(f(h'(A, t), m(A, t))\right) = \omega'(h'(A, t)) \right).
\]
Define a homotopy \( h: \mathcal{H} \times I \to \mathcal{H} \) by
\[
h(A, t) = K_q\left(f(h'(A, t), m(A, t)) \right) \quad \text{for each} \quad A \in \mathcal{H}, \; t \in I.
\]
Clearly, we have \( \omega(h(A, t)) = \omega'(h'(A, t)) \). Hence (4) implies that \( \omega \) satisfies the condition (A2). Obviously, \( \omega \) satisfies the conditions (A1) and (A3). Thus \( \omega \) is strongly admissible.

A metric space \( (X, q) \) is locally hyperconvex if for any \( x \in X \) there is a neighborhood \( U \) of \( x \) in \( X \) such that \( q|U \) is a hyperconvex metric. Then we have the following

(2.2) Theorem. Let \( (X, q) \) be a locally hyperconvex metric continuum and let \( \omega \) be the Whitney map for \( \mathcal{H} = 2^X \) or \( C(X) \) which is defined by (*) and the metric \( q \). Then there exist a positive number \( t_0 \in (0, \omega(X)) \) and a homotopy
\[
h: \omega^{-1}([0, t_0]) \times I \to \omega^{-1}([0, t_0])
\]
such that

\begin{align*}
(A1)' & \quad h(A, 1) = A, \quad h(A, 0) \in F_1(X) \quad \text{for each} \quad A \in \omega^{-1}([0, t_0]); \\
(A2)' & \quad \text{if} \quad \omega(h(A, t)) > 0 \quad \text{for some} \quad A \in \omega^{-1}([0, t_0]) \quad \text{and} \quad t \in I, \quad \text{then} \\
& \quad \omega(h(A, s)) < \omega(h(A, t)) \quad \text{whenever} \quad 0 \leq s < t \leq 1; \\
(A3)' & \quad h(\{x\}, t) = \{x\} \quad \text{for each} \quad x \in X \quad \text{and} \quad t \in I.
\end{align*}

Proof. Since \( q \) is locally hyperconvex, there is a positive number \( \varepsilon > 0 \) such that, for any \( x \in X \), \( q|S(x, \varepsilon) \) is hyperconvex, where
\[
S(x, \varepsilon) = \{y \in X \mid q(x, y) \leq \varepsilon\}.
\]
Also, there are points \( x_1, x_2, \ldots, x_n \) of \( X \) such that
\[
X = \bigcup_{i=1}^{n} \text{Int}S(x_i, \varepsilon/3).
\]
Since \( X \) is an ANR, there is a retraction \( r: \mathcal{U} \to F_1(X) = X \), where \( \mathcal{U} \) is a neighborhood of \( F_1(X) \) in \( \mathcal{H} \). Let \( \delta \) be a positive number such that
\[
(2^n + 1)\delta < \varepsilon/3.
\]
Choose a sufficiently small positive number \( t_0 \in (0, \omega(X)) \) such that
\begin{enumerate}
\item if \( A \in \omega^{-1}([0, t_0]) \), then \( A \subset S(x_i, \varepsilon/3) \) for some \( i \);
\item \( \omega^{-1}([0, t_0]) \subset \mathcal{U} \);
\item if \( A \in \omega^{-1}([0, t_0]) \), then \( S(r(A), \delta) \supset A \).
\end{enumerate}
Set
\[ \mathcal{A}_i = \{ A \in \omega^{-1}(\{0, t_0\}) \mid A \subset S(x_i, \varepsilon/3) \}, \]
\[ \mathcal{B}_i = \{ A \in \omega^{-1}(\{0, t_0\}) \mid A \subset \text{Int } S(x_i, 2\varepsilon/3) \} \quad (i = 1, 2, \ldots, n), \]
\[ \mathcal{A}_0 = \mathcal{B}_0 = \emptyset. \]

Let \( \varphi_i : \omega^{-1}(\{0, t_0\}) \to I \) (\( i = 1, 2, \ldots, n \)) be a map such that

(4) \( \varphi_i(A) = 0 \) if \( A \in \mathcal{A}_i \) and \( \varphi_i(A) = 1 \) if \( A \) is not contained in \( \mathcal{B}_i \).

By induction, we shall construct a homotopy
\[ h_i : \omega^{-1}(\{0, t_0\}) \times I \to \omega^{-1}(\{0, t_0\}) \quad (i = 0, 1, \ldots, n) \]
such that

A (i) \( h_i(A, 1) = A \) for each \( A \in \omega^{-1}(\{0, t_0\}) \);

B (i) if \( A \in \bigcup_{j=0}^i \mathcal{A}_j \), then \( h_i(A, 0) = r(A) \in F_1(X) \);

C (i) if \( A \in \omega^{-1}(\{0, t_0\}) \) and \( 0 \leq s \leq t \leq 1 \), then
\[ \omega(h_i(A, s)) \leq \omega(h_i(A, t)); \]

D (i) \( h_i(\{x\}, t) = \{x\} \) for \( x \in X \) and \( t \in I \);

E (i) if \( \omega(h_i(A, s)) = \omega(h_i(A, t)) \) for some \( A \in \omega^{-1}(\{0, t_0\}) \) and \( s, t \in I \), then \( h_i(A, s) = h_i(A, t) \);

F (i) \( h_i(A, t) \subset S(r(A), 2^i \delta) \) for each \( A \in \omega^{-1}(\{0, t_0\}) \) and \( t \in I \).

First, for the case \( i = 0 \), define
\[ h_0 : \omega^{-1}(\{0, t_0\}) \times I \to \omega^{-1}(\{0, t_0\}) \]
by \( h_0(A, t) = A \) for each \( A \in \omega^{-1}(\{0, t_0\}) \) and \( t \in I \). Next, we suppose that there is a homotopy
\[ h_i : \varphi^{-1}(\{0, t_0\}) \times I \to \omega^{-1}(\{0, t_0\}) \]
satisfying the conditions A (i)–F (i). We shall construct a homotopy
\[ h_{i+1} : \omega^{-1}(\{0, t_0\}) \times I \to \omega^{-1}(\{0, t_0\}) \]
satisfying the conditions A (i+1)–F (i+1). If \( A \in \mathcal{B}_{i+1} \), then (3) implies that \( r(A) \in S(x_{i+1}, (2\varepsilon/3) + \delta) \). By F (i), we can see that if \( A \in \mathcal{B}_{i+1} \), then
\[ h_i(A, 0) \subset S(x_{i+1}, (2\varepsilon/3) + \delta(1 + 2^i)) \subset S(x_{i+1}, \varepsilon). \]
Since \( g(S(x_{i+1}, \varepsilon) \) is hyperconvex, by the proof of (2.1) we have a homotopy
\[ h'_{i+1} : h_i(\mathcal{B}_{i+1} \times \{0\}) \times I \to \omega^{-1}(\{0, t_0\}) \]
such that

(5) \( h'_{i+1}(A, 1) = A \), \( h'_{i+1}(A, 0) = r(A) \) for each \( A \in h_i(\mathcal{B}_{i+1} \times \{0\}); \)
(6) if \( \omega \left( h^i_{i+1} (A, t) \right) > 0 \) for some \( A \in h_i \left( \mathcal{B}_{i+1} \times \{0\} \right) \) and \( t \in I \), then
\[
0 < \omega \left( h^i_{i+1} (A, s) \right) < \omega \left( h^i_{i+1} (A, t) \right) \quad \text{whenever} \quad 0 < s < t \leq 1;
\]

(7) \( h^i_{i+1} (\{x\}, t) = \{x\} \) for \( \{x\} \in h_i \left( \mathcal{B}_{i+1} \times \{0\} \right) \) and \( t \in I \).

Also, by the proof of (2.1) we see that

(8) \( h^i_{i+1} (A, t) \in S(r(A), 2 \sup \{d(r(A), a) \mid a \in A\}) \) for \( A \in h_i \left( \mathcal{B}_{i+1} \times \{0\} \right) \) and \( t \in I \) (because in the proof of (2.1) \( f \) is a contraction).

Define a homotopy

\[
h^i_{i+1} : \omega^{-1}([0, t_0]) \times I \to \omega^{-1}([0, t_0])
\]

by

\[
h^i_{i+1} (A, t) = \begin{cases} 
  h^i (A, 2t - 1) & \text{if } A \in \omega^{-1}([0, t_0]) \text{ and } 1/2 \leq t \leq 1, \\
  h^i_{i+1} (h_i (A, 0), 2t + (1 - 2t) \varphi_{i+1} (A)) & \text{if } A \in \mathcal{B}_{i+1} \text{ and } 0 \leq t \leq 1/2, \\
  h_i (A, 0) & \text{if } A \text{ is not contained in } \mathcal{B}_{i+1} \text{ and } 0 \leq t \leq 1/2.
\end{cases}
\]

By A(i)–E(i), \( h^i_{i+1} \) satisfies the conditions A(i+1)–E(i+1). By F(i) and (8), we see that \( h^i_{i+1} \) satisfies the conditions F(i+1). Thus we have a homotopy

\[
h' = h_n : \omega^{-1}([0, t_0]) \times I \to \omega^{-1}([0, t_0])
\]

such that

(9) \( h'(A, 1) = A \), \( h'(A, 0) = r(A) \in F_1 (X) \) for \( A \in \omega^{-1}([0, t_0]) \);

(10) \( h' (\{x\}, t) = \{x\} \) for \( x \in X \) and \( t \in I \);

(11) if \( \omega (h'(A, s)) = \omega (h'(A, t)) \) for some \( A \in \omega^{-1}([0, t_0]) \) and \( s, t \in I \),

then \( h'(A, s) = h'(A, t) \).

By (9) and (11), we can define a function

\[
h : \omega^{-1}([0, t_0]) \times I \to \omega^{-1}([0, t_0])
\]

by

(12) \( h(A, t) = h'(A, \theta(A, t)) \), where \( \theta(A, t) \) is a positive number such that

\[
\omega (h'(A, \theta(A, t))) = t \cdot \omega (A).
\]

Then \( h \) is continuous. In fact, suppose, on the contrary, that there are a sequence \( A_1, A_2, \ldots \) of points in \( \omega^{-1}([0, t_0]) \) and a sequence \( t_1, t_2, \ldots \) of positive numbers in \( I \) such that

\[
\lim A_n = A \in \omega^{-1}([0, t_0]) \quad \text{and} \quad \lim t_n = t \in I
\]

and

\[
\lim h(A_n, t_n) = B \neq h(A, t).
\]
By (12),
\[
\omega(B) = \lim (t_n \cdot \omega(A_n)) = t \cdot \omega(A) = \omega(h(A, \theta(A, t))).
\]
Note that \( B \in h'(A) \times I \). Hence (11) implies that
\[ B = h'(A, \theta(A, t)) = h(A, t). \]
This is a contradiction. Clearly, \( h \) satisfies the conditions (A1)', (A2)' and (A3)', This completes the proof.

3. Whitney maps of certain polyhedra. In this section, we study Whitney maps of certain polyhedra. Let \( K \) be a cubical complex. Metrize \(|K|\) as follows: Assume that each \( k \)-dimensional cube of \( K \) is a copy of \( I^k \). Define the distance \( \rho \) between two points \( x, y \) of \(|K|\) so that if \( x, y \) are in a common cube \( I^k \), then
\[
\rho(x, y) = \max \{|x_i - y_i| \mid i = 1, 2, \ldots, k\},
\]
where \( x = (x_1, x_2, \ldots, x_k), y = (y_1, y_2, \ldots, y_k) \in I^k \),
otherwise the distance is the length of the shortest path joining them (see [9]). Then \( \rho \) is a convex metric. A connected subset \( Y \) of \(|K|\) is GC (see [9]) if for any cube \( I^k \) of \( K \) either \( Y \cap I^k = \emptyset \) or for some \( 0 \leq s_i \leq t_i \leq 1 \) (\( i = 1, 2, \ldots, k \))
\[
Y \cap I^k = \{(y_1, y_2, \ldots, y_k) \in I^k \mid s_i \leq y_i \leq t_i \ (i = 1, 2, \ldots, k)\}.
\]

A cubical complex \( K \) is regular collapsible [9] if there are a sequence of subcomplexes \( K_0, K_1, \ldots, K_n \) of \( K \) and nonempty subcomplexes \( L_i \) of \( K_i \) such that \( K_0 \) is a one-point set, \( K_n = K \) and
\[
K_{i+1} = K_i \cup (L_i \times I),
\]
where
\[
L_i \times I = \{c \times \{0\}, c \times I, c \times \{1\} \mid c \in L_i\},
\]
and each \(|L_i|\) is GC of \( K_i \). A cubical complex \( K \) is locally regular collapsible if for any \( x \in |K| \) there is a regular collapsible subcomplex \( L \) of \( K \) such that \( x \in \text{Int} |L| \).

In [9], Mai and Tang proved that if \( P \) is a collapsible simplicial polyhedron, then there is a regular collapsible cubical complex \( K \) such that \(|K| = P \). Also, they proved that the metric \( \rho \) as above is a hyperconvex metric. Hence, by (2.1) and the proof of (1.2) (see [6]), we have

(3.1) Theorem. Let \( P \) be a finite collapsible polyhedron. Then there exists a strongly admissible Whitney map \( \omega \) for \( \mathcal{H} = 2^\mathcal{P} \) or \( C(P) \). Moreover, \( \omega|\omega^{-1}((0, \omega(P))) \) is a trivial bundle map with Hilbert cube fibers, where if \( \mathcal{H} = C(P) \), assume that \( P \) contains no free arc.
Also, by (2.2) we have

(3.2) **Theorem.** Let \( K \) be a cubical complex and let \(|K| = P\). If \( K \) is locally regular collapsible, then there is a Whitney map \( \omega \) for \( \mathcal{H} = 2^p \) or \( C(P) \) such that for some \( t_0 \in (0, \omega(P)) \) there is a homotopy

\[
h: \omega^{-1}([0, t_0]) \times I \to \omega^{-1}([0, t_0])
\]

satisfying the conditions (A1)', (A2)' and (A3)' in (2.2). Moreover, \( \omega|\omega^{-1}((0, t_0)) \) is a trivial bundle map with \( P \times Q \) fibers, where if \( \mathcal{H} = C(P) \), assume that \( P \) contains no free arc.

Next, we study Whitney maps of 1-dimensional ANRs.

(3.3) **Lemma.** If \( X \) is a compact 1-dimensional AR (= dendrite), then \( X \) admits a hyperconvex metric. If \( X \) is a compact connected 1-dimensional ANR, then \( X \) admits a locally hyperconvex metric.

**Proof.** Suppose that \( X \) is a compact 1-dimensional AR. By (2.1) in [4] we can conclude that if any collection \( \{A_i\}_{i=1}^n \) of subcontinua of \( X \) satisfies the condition \( A_i \cap A_j \neq \emptyset \), then

\[
\bigcap_{i=1}^n A_i \neq \emptyset.
\]

Since \( X \) is a Peano continuum, \( X \) admits a convex metric \( q \). Then, for any \( x \in X \) and \( \epsilon > 0 \), \( S(x, \epsilon) \) is a subcontinuum of \( X \). Hence we see that \( q \) is hyperconvex. Suppose that \( X \) is a compact 1-dimensional ANR. Let \( q \) be a convex metric on \( X \). By (13.6) in [2], there is a positive number \( \epsilon > 0 \) such that \( S(x, \epsilon) \) is a 1-dimensional AR for each \( x \in X \). Hence \( q|S(x, \epsilon) \) is hyperconvex, which implies that \( q \) is a locally hyperconvex metric.

(3.4) **Lemma.** If \( X_i \) (\( i = 1, 2 \)) admits a hyperconvex metric (resp., locally hyperconvex metric) \( q_i \), then \( X_1 \times X_2 \) admits a hyperconvex metric (resp., locally hyperconvex metric) \( q \), where

\[
q(x, y) = \max \{q_i(x_i, y_i) \mid x = (x_1, x_2), y = (y_1, y_2) \text{ and } i = 1, 2\}.
\]

(3.5) **Corollary.** Let \( X_i \) (\( i = 1, 2, \ldots, n \)) be the \( m(i) \)-sphere (\( m(i) \geq 1 \)) and let \((X_{n+1}, q_{n+1})\) be a locally hyperconvex metric continuum. Suppose that

\[
X = \prod_{i=1}^{n+1} X_i.
\]

Then there is a Whitney map \( \omega \) for \( \mathcal{H} = 2^X \) or \( C(X) \) such that \( \omega|\omega^{-1}((0, t_0)) \) is a trivial bundle map with \( X \times Q \) fibers for some \( t_0 \in (0, \omega(X)) \).

**Outline of proof.** Assume that

\[
X_i = \{x = (x_0, x_1, \ldots, x_{m(i)}) \in \mathbb{R}^{m(i)+1} \mid \|x\| = 1\}.
\]
We define a metric $\varrho_i$ on $X_i$ by

$$\varrho_i(x, y) = \arccos \left[ \sum_{j=0}^{m(i)} x_j y_j \right]$$

for

$$x = (x_0, x_1, \ldots, x_{m(i)}), \ y = (y_0, y_1, \ldots, y_{m(i)}) \in X_i.$$ 

Define a metric $\varrho$ on $X$ by

$$\varrho(x, y) = \max \{ \varrho_i(x_i, y_i) \mid i = 1, 2, \ldots, n+1 \}$$

for

$$x = (x_1, x_2, \ldots, x_{n+1}), \ y = (y_1, y_2, \ldots, y_{n+1}) \in X.$$ 

Let $\omega$ be the Whitney map for $\mathcal{H}$ which is defined by (*) and the metric $\varrho$. By the similar way as in the proofs of (2.1) and (2.2), we can conclude that $\omega$ satisfies the desired conditions.

In the statements of (2.2), (3.2) and (3.5), we cannot conclude that $\tau_0 = \omega(X)$. We have the following

(3.6) Proposition. Let $X$ be a compact connected ANR but not AR. Let $\mathcal{H} = 2^X$ or $C(X)$. If $\mathcal{H} = C(X)$, assume that $X$ contains no free arc. Then for any Whitney map $\omega$ for $\mathcal{H}$ there is no homotopy

$$h: \omega^{-1}([0, \omega(X)]) \times I \to \omega^{-1}([0, \omega(X)])$$

satisfying the conditions (A1)' and (A2)' in (2.2).

Proof. Suppose, on the contrary, that such a homotopy $h$ exists. Since $\mathcal{H}$ is homeomorphic to the Hilbert cube $Q$,

$$\omega^{-1}([0, \omega(X)]) = Q - \{ * \} = Q \times [0, 1].$$

Hence $\omega^{-1}([0, \omega(X)])$ is contractible. Let $f: \omega^{-1}([0, \omega(X)]) \to F_1(X) (= X)$ be a map defined by $f(A) = h(A, 0)$ for each $A \in \omega^{-1}([0, \omega(X)])$ (see (A1)'). Then (A2)' implies that $f|F_1(X) \simeq 1_{F_1(X)}$. Since $X$ is an ANR, by Borsuk's homotopy extension theorem (see [2]), there is a retraction

$$r: \omega^{-1}([0, \omega(X)]) \to F_1(X).$$

Thus $X$ is contractible, and hence $X$ is an AR (see [2]). This is a contradiction.

(3.7) Example. Let $S^1$ be the unit circle in the plane $R^2$ and let $\varrho$ be the arc length metric on $S^1$. Suppose that $\omega$ is the Whitney map for $2^{S^1}$ defined by (*) and the metric $\varrho$. Then $\omega|\omega^{-1}([0, \pi/2])$ is a trivial bundle map with $S^1 \times Q$ fibers, but $\omega|\omega^{-1}([0, \pi/2])$ is not a trivial bundle map; in fact, it is not an open map. Let $A \in \omega^{-1}([0, \pi/2])$. First, we shall show that
there are points \( r_1(A) \) and \( r_2(A) \) of \( A \) such that
\[
q(r_1(A), r_2(A)) < \pi \quad \text{and} \quad A \subset [r_1(A), r_2(A)],
\]
where if \( x, y \in S^1 \), then
\[
[x, y] = \{ z \in S^1 | q(x, z) + q(z, y) = q(x, y) \}.
\]

If \( |A| \leq 2 \) (where \( |A| \) denotes the cardinality of \( A \)), it is easily seen that
\((\#)\) is true. Let \( |A| \geq 3 \). Suppose, on the contrary, that \((\#)\) is not true for
some \( \varnothing \in \omega^{-1}([0, \pi/2]) \). Choose a point \( a \in A \) and let \( a' \) be the point of \( S^1 \)
such that \( q(a, a') = \pi \). Then \( A \) does not contain \( a' \). Let \( S_a \) and \( S_{a'} \) denote the
path components of \( S^1 - \{a, a'\} \). Choose the point \( b \in A \cap S_a \) such that
\( A \cap S_a \subset [a, b] \). Then there is a point \( c \in A \cap S_b \), where \( S_b \) does not contain
\( a \). Note that if \( x \in \{a, b, c\} \), then \( \{x, x'\} \) separates \( S^1 \) between the other two
points \( \{a, b, c\} - \{x\} \). Then we have
\[
\omega(A) \geq \omega(\{a, b, c\}) = (1/2) \max \{q(a, b), q(b, c), q(c, a)\} + (1/4) \min \{q(a, b), q(b, c), q(c, a)\} \geq \pi/2.
\]
This is a contradiction. By \((\#)\) we can easily see that there is a homotopy
\[
h: \omega^{-1}([0, \pi/2]) \times I \rightarrow \omega^{-1}([0, \pi/2])
\]
satisfying the conditions \((A1)', (A2)' \) and \((A3)' \) in (2.2). Hence \( \omega|\omega^{-1}([0, \pi/2]) \)
is a trivial bundle map with \( S^1 \times \mathbb{Q} \) fibers. Let
\[
x_1 = (1, 0), \quad x_2 = (-1/2, \sqrt{3}/2), \quad x_3 = (-1/2, -\sqrt{3}/2)
\]
and let \( A = \{x_1, x_2, x_3\} \). Then \( \omega(A) = \pi/2 \). In [5], (4.15), Goodykoontz and
Nadler pointed out the following fact: there is a neighborhood \( \mathcal{U} \) of \( A \) in \( 2^{S^1} \)
such that if \( B \in \mathcal{U} \), then \( \omega(B) \geq \pi/2 \). This implies that \( \omega|\omega^{-1}([0, \pi/2]) \) is not
an open map.

The following problems remain open:

(i) Let \( X \) be a compact AR. Is there a strongly admissible Whitney map
for \( \mathcal{H} = 2^X \) or \( C(X) \)? (see [6], (3.4)). (P 1358)

(ii) Let \( X \) be a compact ANR (or polyhedron). Is there a Whitney map
\( \omega \) for \( \mathcal{H} = 2^X \) or \( C(X) \) such that for some \( t_0 \in (0, \omega(X)) \) there is a homotopy
\[
h: \omega^{-1}([0, t_0]) \times I \rightarrow \omega^{-1}([0, t_0])
\]
satisfying the conditions \((A1)', (A2)' \) and \((A3)' \) in (2.2)? (P 1359)

REFERENCES


FACULTY OF INTEGRATED ARTS AND SCIENCES
HIROSHIMA UNIVERSITY
HIGASHISENDA-MACHI, NAKA-KU
HIROSHIMA, 730 JAPAN

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