A PROCEDURE REALIZING A SECOND ORDER
ONE-STEP METHOD FOR SOLVING A SYSTEM
OF ORDINARY DIFFERENTIAL EQUATIONS

1. Procedure declaration. The procedure `diffeq2` solves the
initial value problem of the form
\[
\begin{align*}
    y_k &= f_k(x, y_1(x), y_2(x), \ldots, y_n(x)), \\
    y_k(x_0) &= y_{0k} \quad (k = 1, 2, \ldots, n)
\end{align*}
\]
at the points \(x_1, x_2, \ldots\)

Data:
- \(x\) — the value of \(x_0\) in (2);
- \(x1\) — the value of the argument for which we solve the problem;
- \(eps\) — the relative error (the given tolerance);
- \(eta\) — the number which is used instead of zero when the obtained solution is zero or near to zero;
- \(hmin\) — the least admissible absolute value of the step length;
- \(n\) — the number of differential equations in (1) and (2);
- \(y[1:n]\) — the values of the right-hand sides of (2).

Results:
- \(x\) — the value of \(x1\);
- \(y[1:n]\) — the values of the approximate solution \(y_k(x) \quad (k = 1, 2, \ldots, n)\).

Additional parameters:
- \(step\) — label outside of the body of the procedure `diffeq2` to which a jump is made if the absolute value of the step length is smaller than \(hmin\); after the jump, \(x\) is equal to the value \(\tilde{x}\) (\(\tilde{x} < x1\)) for which the approximate solution has a relative error equal to the given \(eps\), and \(y[1:n]\) contains the value of this approximate solution;
- \(f\) — identifier of the procedure which computes the values of the right-hand sides of (1) and puts them in \(d[1:n]\); this procedure has the following heading: procedure \(f(x, n, y, d)\); value \(x, n\); real \(x\); integer \(n\); array \(y, d\).
procedure diffsystbobkov2(x,x1,eps,eta,hmin,n,y,steph,f);
    value x1,eps,eta,hmin,n;
    real x,x1,eps,eta,hmin;
    integer n;
    array y;
    label steph;
    procedure f;
    begin
    real h,hh,w,w1,w2,w3,w4;
    integer i;
    Boolean last;
    array d,df,dff,y1,y2[1:n];
    eps:=.1666666666666666/eps;
    h:=x1-x;
    last:=true;
    f(x,n,y,d);
    conth:
    hh:=h*.5;
    w:=h*.375;
    for i:=1 step 1 until n do
    begin
        w1:=y[i];
        w2:=d[i];
        y1[i]:=w1+hh*w2;
        y2[i]:=w1+w*w2
        end i;
    f(x+hh,n,y1,df);
    f(x+w,n,y2,dff);
    hh:=h*.75;
    for i:=1 step 1 until n do
\textbf{begin}
\begin{align*}
w1 &= y[i]; \\
y1[i] &= w1 + h \times df[i]; \\
y2[i] &= w1 + hh \times dff[i]
\end{align*}
\textbf{end} i;
\begin{align*}
f(x+h, n, y1, df); \\
f(x+hh, n, y2, dff); \\
hh &= h \times 5; \\
w2 &= h \times 333333333333; \\
w &= 0;
\end{align*}
\textbf{for} i = 1 \textbf{ step } 1 \textbf{ until } n \textbf{ do}
\begin{align*}
\textbf{begin}
\begin{align*}
w1 &= d[i]; \\
w3 &= w2\times(w1+2.0\times df[i]); \\
w1 &= w3 - hh\times(w1+df[i]); \\
w3 &= y2[i] = y[i] + w3 + w1 \times 333333333333; \\
w1 &= \text{abs}(w1); \\
w3 &= \text{abs}(w3); \\
\text{if} w3 < \text{eta} \\
\text{then} \ w3 &= \text{eta}; \\
w3 &= w1/w3; \\
\text{if} w3 > w \\
\text{then} \ w &= w3
\end{align*}
\textbf{end} i;
\begin{align*}
w &= \textbf{if} \ w = 0 \textbf{ then } \text{eta} \textbf{ else } 1.25 \times (\text{eps} \times w) \times 333333333333; \\
hh &= h/w;
\end{align*}
\textbf{if} w > 1.25
\begin{align*}
\textbf{then}
\textbf{begin}
\begin{align*}
\text{if} \ \text{abs}(hh) < \text{hmin}
\end{align*}
\textbf{end}
\end{align*}
then go to steph;
last:=false
end w>1.25
else
begin
x:=x+h;
for i:=1 step 1 until n do
y[i]:=y2[i];
if last
then go to endp;
f(x,n,y,d);
w:=x1-x;
if (w-hh)<h<0
then
begin
hh:=w;
last:=true
end (w-hh)<h<0
end w<1.25;
h:=hh;
go to contl;
endp;
end diffysystbobkov2

2. Method used. In the construction of the procedure diffysystbobkov2 we use two one-step methods of second order. The methods are of the form

(3) \[ \eta_{n+1} = \eta_n + \frac{h}{2} (f_n + f_{n+1}) \]

with the additional formulas

\[ \eta_{n+1/2} = \eta_n + \frac{h}{2} f_n, \quad \eta_{n+1} = \eta_n + hf_{n+1/2}, \]
and

\[ \eta_{n+1} = \eta_n + \frac{h}{3} (f_n + 2f_{n+3/4}) \]

with the additional formulas

\[ \eta_{n+3/8} = \eta_n + \frac{3}{8} hf_n, \quad \eta_{n+3/4} = \eta_n + \frac{3}{4} hf_{n+3/4}, \]

where \( \eta_{n+a} \) is an approximate solution obtained at the point \( x_n + ah \), and

\[ f_{n+ah} = f(x_n + ah, \eta_{n+a}). \]

Using the results due to Bobkov [2], p. 38-42, we may treat the solution obtained by (4) as the solution calculated by method (3) twice with step \( h/2 \). In this way, we may use the solutions obtained by methods (3) and (4) to vary the step of integration.

In paper [3] the method (3) is used once with step \( h \) and twice with step \( h/2 \) for the control of the step size.

This step size control mechanism is described in [1]. The numerical results for the procedure \textsc{diffsystbokov2} are better than those of the procedure \textsc{diffsystheun} (see [3]) which realizes the method (3).

3. Certification. The procedure \textsc{diffsystbokov2} has been verified on the ODRA 1204 computer for many examples of the initial value problem. Some of them are presented here.

Examples.

(A) \[ y'_1 = 1/y_2, \quad y_1(0) = 1, \quad y'_2 = -1/y_1, \quad y_2(0) = 1 \]

with the exact solution \( y_1 = e^x, \quad y_2 = e^{-x} \).

(B) \[ y'_1 = 10 \cos 10x, \quad y_1(0) = 0 \]

with the exact solution \( y_1 = \sin 10x \).

(C) \[
\begin{align*}
|y'_1 = 10 \text{sgn}(\sin 20x) y_2, & \quad y_1(0) = 0, \\
|y'_2 = -10 \text{sgn}(\sin 20x) y_1, & \quad y_2(0) = 1,
\end{align*}
\]

with the exact solution \( y_1 = |\sin 10x|, \quad y_2 = |\cos 10x| \).

In the sequel we present the obtained relative error and the number of evaluations of the function \( f (|f|) \).

The results were obtained for \( \text{eps} = \eta \) and \( h_{\text{min}} = 10^{-15} \) at the points \( 0.5, 1.0, 1.5, \) and \( 10.0 \). Here we give only the results for \( x = 1.5 \) and \( x = 10.0 \).
The results obtained for the problem (A)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\text{eps} = 10^{-3}$ [f]</th>
<th>$\text{eps} = 10^{-6}$ [f]</th>
<th>$\text{eps} = 10^{-9}$ [f]</th>
</tr>
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<tbody>
<tr>
<td>1.5</td>
<td>$1.5_{10}^{-5}$ 14</td>
<td>$1.2_{10}^{-7}$ 74</td>
<td>$4.4_{10}^{-10}$ 689</td>
</tr>
<tr>
<td>2.4_{10}^{-4}</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10.0</td>
<td>$-2.8_{10}^{-2}$ 128</td>
<td>$-3.2_{10}^{-5}$ 1173</td>
<td>$-3.2_{10}^{-8}$ 11613</td>
</tr>
<tr>
<td>$3.8_{10}^{-2}$</td>
<td></td>
<td>$4.1_{10}^{-5}$</td>
<td>$4.1_{10}^{-8}$</td>
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The results obtained for the problem (B)

<table>
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<th>$\text{eps} = 10^{-9}$ [f]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>$-7.4_{10}^{-4}$ 117</td>
<td>$5.1_{10}^{-7}$ 728</td>
<td>$4.1_{10}^{-9}$ 6942</td>
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<tr>
<td>10.0</td>
<td>$-5.0_{10}^{-3}$ 2181</td>
<td>$-3.7_{10}^{-6}$ 14217</td>
<td>$9.5_{10}^{-8}$ 134643</td>
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The results obtained for the problem (C)

<table>
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<td>$-5.0_{10}^{-2}$ 15558</td>
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<td>$-5.0_{10}^{-2}$</td>
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References


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