An $S^1$-equivariant degree and the Fuller index

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Abstract. In this paper, we define a homotopy invariant for $S^1$-equivariant maps $\varphi: (U, \partial U) \to (V, V \setminus \{0\})$, where $V$ is a representation of $S^1$ and $U \subset V \oplus R$ is open and bounded. We call this invariant the $S^1$-degree. An infinite-dimensional generalization of the $S^1$-degree is also given. As an application we give a description of the Fuller index in terms of the $S^1$-degree.

Introduction. In 1965 F. B. Fuller introduced an invariant of a flow, called now the Fuller index. Roughly speaking, the Fuller index counts the algebraic number of periodic orbits of flow. It is natural to ask whether the Fuller index can be defined as a homotopy invariant of a map defined by the flow on a function space. This led to a construction of a new homotopy invariant for $S^1$-equivariant maps. We call this invariant the $S^1$-degree despite the fact that it is defined for maps with domain in the $(n+1)$-dimensional euclidean space and range in the $n$-dimensional euclidean space.

In this paper, we first construct the $S^1$-degree in the finite-dimensional case. More precisely, we assume that there is given a pair $(V, g)$, where $V$ is a finite-dimensional linear space over $R$ and $g: S^1 \to GL(V)$ is a continuous homomorphism into the group of all linear automorphisms of $V$. We consider continuous maps $f: X \to V$ such that $X$ is an invariant subset of $V \oplus R$ (i.e. $(x, \lambda) \in X$, $g \in S^1$ imply $(g(x), x, \lambda) \in X$) and $f$ is equivariant (i.e. $f(g(x), x, \lambda) = g(g)f(x, \lambda)$ for all $g \in S^1$, $(x, \lambda) \in X$). Suppose $\Omega \subset V \oplus R$ is bounded, invariant, $\Omega \subset X$ and $f(x, \lambda) \neq 0$ for $(x, \lambda) \in \partial \Omega$; then we define $\text{Deg}(f, \Omega) = \{a_r\}$, where $r \in \{0\} \cup \mathbb{N}$, $a_0 \in \mathbb{Z}_2$, $a_r \in \mathbb{Z}$, $a_r = 0$ for almost all $r$; $\mathbb{Z}$ denotes the group of integers and $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

In order to give at least a very rough idea of our approach we will sketch the definition of $\text{Deg}(f, \Omega)$ for $f$ satisfying additional assumptions A.1–A.3 below. In some sense, equivariant maps $f$ satisfying these assumptions are "generic" for our construction. First we assume

A.1. There exists $a \in \Omega$ such that $f^{-1}(0) = \{g(a): g \in S^1\}$.

Recall that the subgroup of $S^1$ defined by $G_a = \{g \in S^1: g(a) = a\}$ is called the isotropy group of $a$. Since $g$ is continuous, $G_a$ is either finite or $S^1$. Our second assumption is
A.2. \( G_a \) is finite.

Note that since \( G_a \) is a finite subgroup of \( S^1 \) there exists \( k \in \mathbb{N} \) such that \( G_a = \mathbb{Z}_k \), the subgroup of \( S^1 \) consisting of \( k \)-th roots of 1. Without loss of generality we may assume that \( (V, g) \) is orthogonal, i.e. all \( g \in G_a \) are orthogonal with respect to a scalar product in \( V \). Let \( W = \{ x \in V : g(x) = x \text{ for } g \in G_a \} \). Take the direct sum decomposition \( V = W \oplus W^\perp \). Writing points of \( V \oplus R = W \oplus W^\perp \oplus R \) in the form \( (x, y, \lambda) \), we have

\[
  f(x, y, \lambda) = (f_1(x, y, \lambda), f_2(x, y, \lambda)) \in W \oplus W^\perp.
\]

Our last additional assumption is

A.3. \( f_2(x, y, \lambda) = y \) for all \( (x, y, \lambda) \in \Omega \).

Now assume that \( D^p, p = \text{the dimension of } W \), is a sufficiently small closed disc contained in \( \Omega \cap (W \oplus R) \) and transversal to the orbit \( G_a = M \) at \( a \). Orienting \( D^p \) in a suitable way we identify it with the standard unit disc in \( W \). Then, using the classical Brouwer degree we define an integer \( \deg(\varphi, D^p) \), where \( \varphi \) denotes the restriction of \( f \). Finally, we let \( \text{Deg}(f, \Omega) = \{ \alpha_r \} \), where \( \alpha_k = \deg(\varphi, D^p) \) and \( \alpha_r = 0 \) for \( r \neq k \).

In Section 1 we introduce notations, recall basic facts concerning finite-dimensional representations of \( S^1 \) and formulate the main theorem (Theorem 1.2) which lists the most important properties of our degree. Sections 2 and 3 are devoted to the proof of Theorem 1.2. First, we construct our degree under the assumption that the set \( \Omega \) consists of points with the same isotropy group (Section 2). For maps of an arbitrary \( \Omega \), we define the \( S^1 \)-degree in Section 3. In Section 4 we discuss the case of differentiable equivariant maps; assuming that 0 is a regular value of \( f \) we derive formulas for the \( S^1 \)-degree. In Sections 5 and 6, following the classical Leray–Schauder theory, we consider an infinite-dimensional generalization of our degree. In Sections 7 and 8 we discuss the relationship between our degree and the Fuller index.

The idea of defining a (generalized) degree for \( S^1 \)-equivariant maps is not new. In [6] Dancer constructs such a degree for \( S^1 \)-equivariant gradient maps. The relationship between these two degrees will be discussed in a separate paper [10] by Dylawerski. Another version of \( S^1 \)-equivariant degree is defined in Ize–Massabó–Vignoli [14] (the authors use cohomological obstruction theory).

It should also be noted that the methods used in the paper closely relate it to the work of Rubinsztein [19] and tom Dieck [7].

1. Degree for \( S^1 \)-equivariant maps. For \( r > 0 \) we let

\[
  D^n(r) = \{ x \in R^n : |x| \leq r \}, \quad S^{n-1}(r) = \{ x \in R^n : |x| = r \},
\]

\[
  \mathcal{R}^n(r) = \{ x \in R^n : |x| < r \}.
\]

We also let

\[
  D^n = D^n(1), \quad S^{n-1} = S^{n-1}(1), \quad B^n = B^n(1).
\]
For \( m \in \mathbb{N} \), we let \( Z_m = \mathbb{Z}/m\mathbb{Z} \); we often use the identification

\[
Z_m = \{ g \in S^1 : \quad g = e^{i\theta}, \quad \theta = 2\pi j/m, \quad j = 0, 1, \ldots, m-1 \}.
\]

We begin by recalling some terminology and facts concerning group actions and group representations. Let \( G \) denote a compact Lie group and let \( e \in G \) be its neutral element.

We say that \( G \) acts on a topological space \( X \) (or \( X \) is a \( G \)-space) if there is a continuous map \( \mu : G \times X \to X \) such that

(i) \( \mu(e, x) = x \),

(ii) \( \mu(g_1, \mu(g_2, x)) = \mu(g_1g_2, x) \),

for all \( g_1, g_2 \in G \) and \( x \in X \). In what follows, for simplicity of notation, we write \( gx = \mu(g, x) \). For a given \( x \in X \) the subgroup \( G_x = \{ g \in G, \quad gx = x \} \) is called the isotropy group of \( x \) and the set \( Gx = \{ gx, \quad g \in G \} \) is called the orbit of \( x \). Two points \( x, y \in X \) are of the same orbit type if there exists \( g \in G \) such that \( G_x = g^{-1}G_yg \).

Given a subgroup \( H \subseteq G \) we let

\[
X^H = \{ x \in X : \quad hx = x \quad \text{for all} \quad h \in H \} = \{ x \in X : \quad H \subseteq G_x \},
\]

\[
X_H = \{ x \in X : \quad G_x = H \}.
\]

Let \( X, Y \) be two \( G \)-spaces. We say that a map \( f : X \to Y \) is \( G \)-equivariant if \( f(gx) = gf(x) \) for all \( x \in X, \quad g \in G \). We say that \( f \) is a \( G \)-map if it is equivariant and continuous. We say that a continuous map \( h : X \times [0,1] \to Y \) is a \( G \)-homotopy between two \( G \)-maps \( f_0, f_1 : X \to Y \) if \( h(x,0) = f_0(x) \), \( h(x,1) = f_1(x) \) and \( h(gx, t) = gh(x, t) \) for all \( g \in G, \quad x \in X, \quad t \in [0,1] \). The set of all \( G \)-maps from \( X \) into \( Y \) will be denoted by \( \text{Map}_G(X,Y) \). We also let \([X,Y]_G \) denote the set of all \( G \)-homotopy classes of \( G \)-maps from \( X \) into \( Y \).

A representation of \( G \) is a pair \( V = (V_0, \varrho) \), where \( V_0 \) is a finite-dimensional, real linear space and \( \varrho : G \to \text{GL}(V_0) \) is a continuous homomorphism from \( G \) into the group of all linear automorphisms of \( V_0 \). Note that if \( V = (V_0, \varrho) \) is a representation of \( G \), then letting \( gv = \varrho(g)(v) \) we obtain a \( G \)-action of \( G \). Moreover, \( g(\alpha v + \beta w) = \alpha(gv) + \beta(gw) \) for all \( g \in G, \quad v, w \in V_0 \) and \( \alpha, \beta \in \mathbb{R} \), i.e. the action is linear. It is evident that any linear \( G \)-action on \( V_0 \) defines a representation. We often do not distinguish between \( V \) and \( V_0 \) using the same letter for a representation and the underlying linear space. Two representations \( V \) and \( W \) of \( G \) are equivalent if there exists an equivariant linear isomorphism \( T : V \to W \). Given two representations \( V \) and \( W \) of \( G \) we denote by \( V \oplus W \) the direct sum of \( V \) and \( W \), i.e. the direct sum of linear spaces with the linear group action defined by \( g(v,w) = (gv, gw) \).

Throughout the rest of the paper we assume \( G = S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \).

For \( m \in \mathbb{N} \) define \( q^m : S^1 \to \text{GL}(2, \mathbb{R}) \) by

\[
q^m(e^{i\theta}) = \begin{bmatrix} \cos m\theta & -\sin m\theta \\ \sin m\theta & \cos m\theta \end{bmatrix}, \quad 0 \leq \theta \leq 2\pi.
\]
For \( k, m \in \mathbb{N} \) we denote by \( R[k, m] \) the direct sum of \( k \) copies of \((\mathbb{R}^2, \mathcal{S}^m)\); we also denote by \( R[k, 0] \) the trivial \( k \)-dimensional representation of \( S^1 \). The following classical result gives a complete classification up to equivalence of finite-dimensional representations of \( S^1 \) (Adams [1]).

**1.1. Theorem.** If \( V \) is a representation of \( S^1 \) then there exist finite sequences \( \{k_i\}, \{m_i\} \) satisfying
\[
(*) \quad m_i \in \{0\} \cup \mathbb{N}, \quad k_i \in \mathbb{N}, \quad 1 \leq i \leq r, \quad m_1 < \ldots < m_r,
\]
such that \( V \) is equivalent to \( \bigoplus R[k_i, m_i] \). Moreover, the equivalence class of \( V \) is uniquely determined by \( \{m_i\}, \{k_i\} \) satisfying (*).

Suppose now that \( V \) is a representation of \( S^1 \). Let \( \Omega \) be an open bounded invariant subset of \( V \oplus \mathbb{R} \). Suppose further that \( f: \Omega \to V \) is a \( G \)-map such that \( f(\partial \Omega) \subset V \setminus \{0\} \). Our construction, carried over in Sections 2 and 3, assigns to each closed subgroup \( H \subset S^1 \) an element
\[
\deg_H(f, \Omega) \in \begin{cases} \mathbb{Z} & \text{if } H \text{ is finite,} \\ \mathbb{Z}_2 & \text{if } H = S^1. \end{cases}
\]
We denote by \( \mathcal{A}_0 \) the free abelian group generated by \( \mathcal{A} \) and let \( \mathcal{A} = \mathbb{Z}_2 \oplus \mathcal{A}_0 \). Note that \( \alpha \in \mathcal{A} \) means \( \alpha = \{\alpha_r\} \), where \( \alpha_0 \in \mathbb{Z}_2 \) and \( \alpha_r \in \mathbb{Z} \) for \( r \in \mathbb{N} \), and \( \alpha_r = 0 \) for almost all \( r \). Using the fact that there is a one-to-one correspondence between \( \mathcal{A} \) and the family of all proper closed subgroups of \( S^1 \), we define
\[
\text{Deg}(f, \Omega) = \{\alpha_r\} \in \mathcal{A} \text{ by } \alpha_0 = \deg_0(f, \Omega) \text{ and } \alpha_r = \deg_r(f, \Omega) \text{ if } r = |H|.
\]
We are now in a position to state our main result.

**1.2. Theorem.** Let \( V \) run through representations of \( S^1 \), \( \Omega \) through the family of all open bounded invariant subsets of \( V \oplus \mathbb{R} \), and \( f: X \to V \) through \( G \)-maps such that \( X \) is invariant, \( \Omega \subset X \) and \( f(\partial \Omega) \subset V \setminus \{0\} \). Then there exists an \( \mathcal{A} \)-valued function \( \text{Deg}(f, \Omega) = \{\deg_H(f, \Omega)\} \), called the \( S^1 \)-degree, satisfying the following conditions:

(a) If \( \deg_H(f, \Omega) \neq 0 \) then \( f^{-1}(0) \cap \Omega \neq \emptyset \).

(b) If \( \Omega_0 \subset \Omega \) is open, invariant and \( f^{-1}(0) \cap \Omega \subset \Omega_0 \) then \( \text{Deg}(f, \Omega) = \text{Deg}(f, \Omega_0) \).

(c) If \( \Omega_1, \Omega_2 \) are open invariant subsets of \( \Omega \) such that \( \Omega_1 \cap \Omega_2 = \emptyset \) and \( f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2 \) then \( \text{Deg}(f, \Omega) = \text{Deg}(f, \Omega_1) + \text{Deg}(f, \Omega_2) \).

(d) If \( h: (\Omega_0 \times [0, 1], \partial \Omega_0 \times [0, 1]) \to (V, V \setminus \{0\}) \) is a \( G \)-homotopy then \( \text{Deg}(h_0, \Omega) = \text{Deg}(h_1, \Omega) \).

(e) Suppose \( W \) is another representation of \( G = S^1 \) and let \( U \) be an open bounded invariant subset of \( W \) such that \( 0 \in U \). Define \( F: U \times \Omega \to W \oplus V \) by \( F(x, y) = (x, f(y)) \). Then \( \text{Deg}(F, \Omega) = \text{Deg}(f, \Omega) \).

**2. Proof of a special case of Theorem 1.2.** First we recall basic properties of the topological (Brouwer) degree in \( \mathbb{R}^n \)(cf. Amann–Weiss [2], Chow–Hale [4], Dugundji–Granas [9]).
Let $U$ be an open bounded subset of $\mathbb{R}^n$. If $f: X \to \mathbb{R}^n$ is a continuous map such that $\bar{U} \subset X$ and $f(\partial U) \subset \mathbb{R}^n \setminus \{0\}$ then there is defined an integer $\text{deg}(f, U)$, called the degree of $f$ with respect to $U$. All the constructions and results of Sections 2, 3 and 4 are based on the following properties of degree.

(I) (Existence of solutions). If $\text{deg}(f, U) \neq 0$ then there exists $x_0 \in U$ such that $f(x_0) = 0$.

(II) (Excision). If $U_0$ is an open subset of $\mathbb{R}^n$ such that $U_0 \subset U$ and $f^{-1}(0) \cap U = f^{-1}(0) \cap U_0$ then $\text{deg}(f, U_0) = \text{deg}(f, U)$.

(III) (Additivity). If $U_1, U_2$ are open disjoint subsets of $U$ such that $f^{-1}(0) \cap U = U_1 \cup U_2$ then $\text{deg}(f, U) = \text{deg}(f, U_1) + \text{deg}(f, U_2)$.

(IV) (Homotopy). If $h: U \times [0, 1] \to \mathbb{R}^n$ is a continuous map such that $h^{-1}(0) \subset U \times [0, 1]$ then $\text{deg}(h_0, U) = \text{deg}(h_1, U)$, where $h_i(x) = h(x, i)$ for $x \in U$, $i = 0, 1$.

(V) (Contraction). Let $W = U \times (-1, 1) \subset \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1}$ and define $F: W \to \mathbb{R}^{n+1}$ by $F(x, t) = (f(x), t)$. Then $\text{deg}(F, W) = \text{deg}(f, U)$.

(VI) (Diffeomorphism invariance). Suppose $U_0$ is an open subset of $\mathbb{R}^n$ and $d: U_0 \to \mathbb{R}^n$ is an orientation preserving diffeomorphism such that $U = d(U_0)$. Then $\text{deg}(f \circ d, d^{-1}(U)) = \text{deg}(f, U)$.

Note that (I)–(IV) are the well-known fundamental properties of degree (see Amann–Weiss [2]). Properties (V) and (VI) are easy consequences of the differential interpretation of degree (see Chow–Hale [4]).

For the rest of this section we fix an orthogonal representation $V$ of $G = S^1$. We let $n$ be the dimension of the real space $V$. If the action of $G$ is not involved we do not distinguish between $V$ and $\mathbb{R}^n$. For an invariant open bounded subset $\Omega \subset V \oplus \mathbb{R}$ we denote by $C_G(\Omega)$ the linear normed space of all $G$-maps $f: \Omega \to V$ with the standard sup norm. If $X$ is a closed invariant subset of $\Omega$ we let

$$C_G(\Omega, X) = \{f \in C_G(\Omega): f(X) \subset V \setminus \{0\}\}.$$

Clearly $C_G(\Omega, X)$ is an open subset of $C_G(\Omega)$. We say that $f_0, f_1 \in C_G(\Omega, X)$ are $G$-homotopic in $C_G(\Omega, X)$ if there exists a $G$-homotopy $h: \Omega \times [0, 1] \to V$ such that $h(X \times [0, 1]) \subset V \setminus \{0\}$ and $h_0 = f_0$, $h_1 = f_1$.

In order to make our proofs more readable, we introduce some notational conventions. Suppose $W$ is a finite-dimensional linear space over $\mathbb{R}$ and $A: W \to W$ is a linear automorphism. Choose an orientation of $W$ and let

$$\text{sgn } A = \begin{cases} +1 & \text{if } A \text{ preserves orientation}, \\ -1 & \text{if } A \text{ reverses orientation}, \end{cases}$$

Note that $\text{sgn } A$ does not depend on the choice of orientation of $W$.

If $\Phi: W \oplus \mathbb{R} \to W$ is a linear map such that $\Phi(W \oplus \mathbb{R}) = W$ then $\text{Ker } \Phi$ is a one-dimensional linear subspace of $W \oplus \mathbb{R}$. If $v \in \text{Ker } \Phi$ and $v \neq 0$ choose
a linear functional $\varphi: W \oplus R \rightarrow R$ such that $\varphi(v) = 1$. Define $A: W \oplus R \rightarrow W \oplus R$ by $A(x) = (\Phi(x), \varphi(x))$. Evidently $A$ is an automorphism. We let

$$\text{sgn}(\Phi, v) = \text{sgn} A;$$

clearly this definition is correct, i.e. independent of the choice of $\varphi$.

Suppose $a \in V \oplus R$ and $G_a \neq S^1$. Then $M = Ga$ is a one-dimensional submanifold of $V \oplus R$. Set $v = \eta'(0)$, where $\eta(t) = g(t) \cdot a, g(t) = \exp(2\pi it)$. Thus $v$ is tangent to $M$ at $a$ and determines an orientation of $M$. Let

$$N = \{x \in V \oplus R: \langle x, v \rangle = 0\};$$

we call $N$ the normal space at $a$. Note that $a \in N$. We say that a map $A: N \rightarrow V$ is a normal map at $a$ if it is a linear isomorphism and $\text{sgn}(\Phi_A, v) = 1$, where $\Phi_A = A \circ P_N$ and $P_N$ denotes the orthogonal projection of $V \oplus R$ onto $N$.

The following proposition is an immediate consequence of the Slice Theorem (see Bredon [3]).

2.1. Proposition. Let $B_N(a, \varepsilon) = \{x \in N: |x - a| < \varepsilon\}$ and let $U = GB_N(a, \varepsilon)$. There exists $\varepsilon_0$ such that if $\varepsilon < \varepsilon_0$ then $U$ is a tube about $Ga$ and $B_N(a, \varepsilon)$ is a slice for $U$, and consequently satisfies the following condition:

$$(*) \quad x = g_1 y_1 = g_2 y_2, \quad x \in U, \quad y_1, y_2 \in B_N(a, \varepsilon), \quad g_1, g_2 \in G$$

imply $g_1 (g_2)^{-1} \in G_a$.

2.2. Definition. Let $a \in V \oplus R$ and $G_a \neq G$. We say that a continuous map $\varphi: B^n(2) \rightarrow V \oplus R$ is a slice map at $a$ if

(a) $\varphi(0) = a$,

(b) there exists $\varepsilon > 0$ such that $B_N(a, \varepsilon)$ is a slice and $\varphi(B^n(2)) \subset B_N(a, \varepsilon)$,

(c) there exists a normal map at $a$, $A: N \rightarrow V$, such that $\varphi(x) = a + A^{-1}(x)$.

Denote by $H = \{H_1, \ldots, H_k\}$ the family of all closed subgroups of $S^1$ such that $V_H \neq \emptyset$ and let $K = H_1 \cap \ldots \cap H_k$, $k = |K|$, i.e. $K$ is the smallest isotropy group of $V$. First we assume $K \neq S^1$ and postpone the case $K = S^1$ to the end of this section.

2.3. Assumption. Throughout the rest of this section we assume that $\Omega \subset V \oplus R$ is an invariant open bounded set and $(\Omega)_K = \emptyset$.

Let $C(D^n)$ be the linear normed space of all continuous maps $f: D^n \rightarrow R^n$ with the sup norm and let

$$C(D^n, S^{n-1}) = \{f \in C(D^n): f(S^{n-1}) \subset R^n \setminus \{0\}\}.$$ 

We will later need the following lemma.

2.4. Lemma. $C(D^n, S^{n-1})$ is an open dense subset of $C(D^n)$.

Proof. Clearly $C(D^n, S^{n-1})$ is open in $C(D^n)$. To prove it is dense, choose $f \in C(D^n)$ and $\varepsilon > 0$. It is well known that the set of $C^1$-mappings is dense in
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C(D$^n$). Hence there exists $f_0 \in C(D^n)$ such that $\|f - f_0\| < \varepsilon$ and $f_0$ extends to a $C^1$-map $F: U \to \mathbb{R}^n$, where $U$ is an open subset containing $D^n$. Using Sard’s theorem choose a regular value $y_0 \in \mathbb{R}^n$ of $F$ such that $|y_0| < \varepsilon$ and define $F_1: U \to \mathbb{R}^n$ by $F_1(x) = F(x) - y_0$. Choose $r_0 > 1$ such that $D^n(r_0) \subset U$. Since $0$ is a regular value of $F_1$, there exists $r$, $1 \leq r < r_0$, such that $|F_1(rx) - F_1(x)| < \varepsilon$ for all $x \in D^n$ and $F_1^{-1}(0) \cap S^{n-1}(r) = \emptyset$. Define $f_1$ by $f_1(x) = f_1(rx)$. Then $f_1^{-1}(0) \cap S^{n-1} = \emptyset$, $\|f_1 - f\| < 3\varepsilon$ and the conclusion follows.

We frequently use the following simple observation.

2.5. Lemma. Suppose $f_0 \in C_0(\bar{\Omega}, X)$ and let $\varepsilon = \inf \{|f_0(x)|: x \in X\}$. If $f \in C_0(\bar{\Omega})$ and $\|f - f_0\| < \varepsilon$ then $f \in C_0(\bar{\Omega}, X)$ and $f$ is $G$-homotopic to $f_0$ in $C_0(\bar{\Omega}, X)$.

Proof. Clearly $f \in C_0(\bar{\Omega}, X)$. The formula $h(x, t) = (1-t)f_0(x) + tf(x)$ defines the required homotopy.

2.6. Lemma. Let $\varphi: B^n(2) \to \Omega$ be a slice map. If $F \in C(D^n)$ then there exists $f \in C_0(\bar{\Omega})$ such that $\|f\| = \|F\|$ and $F(x) = f(\varphi(x))$ for all $x \in D^n$.

Proof. Extend $F$ to a continuous map $F_1: B^n(2) \to \mathbb{R}^n$ such that $F_1(x) = 0$ outside a compact subset of $B^n(2)$ and $\sup \{|F_1(x)|: x \in B^n(2)\} = \|F\|$.

Since $\varphi$ is a slice map there exists $B_N(a, \varepsilon)$ such that $\varphi(B^n(2)) \subset B_N(a, \varepsilon)$. Set $U = G\varphi(B^n(2))$. If $x \in U$ then there exists $y \in G = S^1$ such that $yx \in B_N(a, \varepsilon)$, and we let $f(x) = g^{-1}F_1(\varphi^{-1}(yx))$; if $x \in \bar{\Omega} \setminus U$ we let $f(x) = 0$. In view of 2.1 and 2.2 it is easy to see that $f$ is the required map.

2.7. Definition. We say that an open invariant subset $\Omega_0 \subset \Omega$ is elementary if there exists a finite family $\{\Omega_1, \ldots, \Omega_r\}$ of open invariant subsets of $\Omega$ such that

\begin{enumerate}
  \item[(*)] $\Omega_0 \subset \Omega_1 \cup \ldots \cup \Omega_r$,
  \item[(**)] $\Omega_i \cap \Omega_j = \emptyset$ \quad for $i \neq j$,
  \item[(***)] for each $i$, $1 \leq i \leq r$, there is a slice map $\varphi_i: B^n(2) \to \Omega$ such that $\Omega_i \subset G\varphi_i(B^n)$.
\end{enumerate}

We call $f \in C_0(\bar{\Omega}, \partial \Omega)$ an elementary $G$-map if there exists an elementary subset $\Omega_0 \subset \Omega$ such that $f^{-1}(0) \subset \Omega_0$. Similarly we say that a $G$-equivariant homotopy $h: (\bar{\Omega}, \partial \Omega) \to (V, V \setminus \{0\})$ is elementary if there exists an elementary subset $\Omega_0 \subset \Omega$ such that $h^{-1}(0) \subset \Omega_0 \times [0, 1]$.

2.8. Lemma. If $f_0 \in C_0(\bar{\Omega}, \partial \Omega)$ is an elementary $G$-map then there exists $\varepsilon > 0$ such that $f \in C_0(\bar{\Omega}, \partial \Omega)$ and $\|f_0 - f\| < \varepsilon$ imply that $f$ is elementary and $h$ defined by $h(x, t) = (1-t)f_0(x) + tf(x)$ is an elementary $G$-homotopy between $f_0$ and $f$.

Proof. Let $\Omega_0$ be an elementary subset of $\Omega$ such that $f_0^{-1}(0) \subset \Omega_0$ and let $\varepsilon = \inf \{|f_0(x)|: x \in \Omega \setminus \Omega_0\}$. It is evident that $\varepsilon$ has the required property.
Assume now that \( f \in C_\varnothing(\varnothing, \partial \varnothing) \) is an elementary \( G \)-map and \( \{\Omega_i\}, \{\varphi_i\} \) satisfy the conditions of Definition 2.7. Let \( U_i = \varphi_i^{-1}(\Omega_i) \) and \( F_i = f \circ \varphi_i; U_i \rightarrow \mathbb{R}^n \). Clearly \( F_i^{-1}(0) \) is a compact subset of \( U_i \) and thus \( \text{deg}(F_i, U_i) \) is well defined for \( i = 1, \ldots, r \). Define

\[
\text{deg}_K(f, \Omega) = \text{deg}(F_1, U_1) + \ldots + \text{deg}(F_r, U_r).
\]

From properties (I)-(VI) of the classical topological degree it follows at once that the definition of \( \text{deg}_K(f, \Omega) \) is independent of the choice of \( \{\Omega_i\} \) and \( \{\varphi_i\} \).

The following observation is a direct consequence of our definition of \( \text{deg}_K(f, \Omega) \).

2.9. Remark. If \( h_i: (\varnothing, \partial \varnothing) \rightarrow (V, V \setminus \{0\}) \) is an elementary \( G \)-homotopy then \( \text{deg}_K(h_0, \Omega) = \text{deg}_K(h_1, \Omega) \).

2.10. Lemma. Let \( \varphi: B^n(2) \rightarrow \varnothing \) be a slice map and let \( \Omega_0 = G\varphi(B^n) \). Then \( C_\varnothing(\varnothing, \partial \varnothing \cup \partial \varnothing_0) \) is a dense subset of \( C_\varnothing(\varnothing, \partial \varnothing) \).

Proof. Suppose \( f \in C_\varnothing(\varnothing, \partial \varnothing) \) and choose \( 0 < \varepsilon < \inf\{|f(x)|: x \in \partial \varnothing\} \). Let \( F \) denote the map defined by \( F(x) = f(\varphi(x)) \); clearly \( F \in C(D^n) \). By 2.4 there exists \( F_i \in C(D^n, S^{n-1}) \) such that \( ||F - F_i|| < \varepsilon \) in \( C(D^n) \). By 2.6 there exists \( f_0 \in C_\varnothing(\varnothing, \partial \varnothing) \) such that \( ||f_0|| < \varepsilon \) and \( f_0(\varphi(x)) = F(x) - F_i(x) \) for all \( x \in D^n \). Let \( f_1(x) = f(x) - f_0(x) \). Since \( ||f_1|| = ||f_0|| < \varepsilon \), \( f_1 \in C_\varnothing(\varnothing, \partial \varnothing \cup \partial \varnothing_0) \). Moreover, \( f_1(\varphi(x)) = F_1(x) \) for all \( x \in D^n \); hence \( f_1 \in C_\varnothing(\varnothing, \partial \varnothing \cup \partial \varnothing_0) \). Therefore \( C_\varnothing(\varnothing, \partial \varnothing \cup \partial \varnothing_0) \) is dense in \( C_\varnothing(\varnothing, \partial \varnothing) \) and the proof is complete.

To define our degree in the general case we need the following two results.

2.11. Proposition. The set of all elementary \( G \)-maps is an open and dense subset of \( C_\varnothing(\varnothing, \partial \varnothing) \).

Proof. From 2.8 it follows that the set of all elementary \( G \)-maps is an open subset of \( C_\varnothing(\varnothing, \partial \varnothing) \). To prove it is dense assume \( f_0 \in C_\varnothing(\varnothing, \partial \varnothing) \). There exists a finite family of slice maps \( \varphi_i: B^n(2) \rightarrow \varnothing, i = 1, \ldots, r \), such that \( \{G\varphi_i(B^n)\} \) covers \( f_0^{-1}(0) \). Let \( U_i = G\varphi_i(B^n) \). Take \( \varepsilon \) such that \( 0 < \varepsilon < \inf\{|f_0(x)|: x \in \varnothing \setminus (U_1 \cup \ldots \cup U_r)\} \). Since an intersection of open and dense subsets is open and dense, Lemma 2.10 implies that \( C_\varnothing(\varnothing, \partial \varnothing \cup \partial U_1 \cup \ldots \cup \partial U_r) \) is an open and dense subset of \( C_\varnothing(\varnothing, \partial \varnothing) \). Therefore there exists \( f \in C_\varnothing(\varnothing, \partial \varnothing \cup \partial U_1 \cup \ldots \cup \partial U_r) \) such that \( ||f_0 - f|| < \varepsilon \). Let \( \Omega_i = U_i \setminus (U_1 \cup \ldots \cup U_{i-1}) \), \( \Omega_i = U_1 \). It is evident that the \( \Omega_i \) are mutually disjoint and \( f_i^{-1}(0) \subset \Omega_1 \cup \ldots \cup \Omega_r \). Thus the proof is complete.

2.12. Proposition. Suppose \( f_0, f_1 \in C_\varnothing(\varnothing, \partial \varnothing) \) are two elementary \( G \)-maps which are \( G \)-homotopic in \( C_\varnothing(\varnothing, \partial \varnothing) \). Then \( \text{deg}_K(f_0, \Omega) = \text{deg}_K(f_1, \Omega) \).

The proof of 2.12 will be based on the following technical lemma.

2.13. Lemma. Suppose \( h: (\varnothing \times [0, 1], \partial \varnothing \times [0, 1]) \rightarrow (V, V \setminus \{0\}) \) is a \( G \)-homotopy and \( \{U_i\}, 1 \leq i \leq r \), is a finite family of open invariant subsets of \( \varnothing \) such that
(\ast) \ h^{-1}(0) \subset (U_1 \cup \ldots \cup U_r) \times [0, 1),

(\ast\ast) \ there \ exist \ slice \ maps \ \phi_i: \ B^n(2) \to \Omega, \ such \ that \ U_i \subset G \phi_i(B^n) \ for \ i = 1, \ldots, r,

(\ast\ast\ast) \ h_0, h_1 \in C_0(\overline{\Omega}, \partial \Omega \cup \partial U_1 \cup \ldots \cup \partial U_r) \ are \ elementary.

Then \ \deg_K(h_0, \Omega) = \deg_K(h_1, \Omega).

Proof. \ The \ proof \ is \ by \ induction \ on \ r. \ Property \ (IV) \ of \ the \ classical \ degree \ implies \ that \ the \ lemma \ is \ true \ for \ r = 1. \ Suppose \ now \ it \ is \ true \ for \ r - 1 \ and \ let \ X = \Pi(h^{-1}(0)), \ where \ \Pi: \ \overline{\Omega} \times [0, 1] \to \overline{\Omega} \ denotes \ the \ projection. \ Clearly \ X \ is \ a \ compact \ subset \ of \ U_1 \cup \ldots \cup U_r. \ Let \ U = U_2 \cup \ldots \cup U_r, \ \Omega_0 = U_1 \setminus \overline{\Omega} \ and \ \Omega_1 = U \setminus \overline{U_1}. \ Let \ \theta: \ \overline{\Omega} \to [0, 1] \ be \ a \ continuous \ function \ such \ that

(i) \ \theta(gx) = \theta(x) \ for \ all \ x \in \overline{\Omega}, \ g \in G,

(ii) \ \theta(x) = 1 \ for \ x \in X \cap \overline{\Omega}_0,

(iii) \ \theta(x) = 0 \ for \ x \in \overline{\Omega} \setminus U_1.

For \ (x, t) \in \overline{\Omega} \times [0, 1] \ set \ \ h^{(1)}(x, t) = h(x, \theta(x) \cdot t), \ h^{(2)}(x, t) = h(x, \theta(x) + t(1 - \theta(x))) \ and \ f(x) = h^{(1)}(x, 1) = h^{(2)}(x, 0). \ From \ the \ definition \ of \ \theta \ \ it \ follows \ that \ f \ is \ elementary \ and \ f \in C_0(\overline{\Omega}, \partial \Omega \cup \partial \Omega_0 \cup \partial \Omega_1). \ Since \ h^{(1)} \ determines \ an \ elementary \ G-homotopy \ between \ h_0 \ and \ f \ in \ C_0(\overline{\Omega}, \overline{\Omega} \setminus (U_1 \cup \Omega_1)) \ \ and \ \ U_1 \cap \Omega_1 = \emptyset, \ Remark \ 2.9 \ implies \ \deg_K(h_0, U_1) = \deg_K(f, U_1).

Moreover, \ h^{(1)}(x, t) = f(x) = h_0(x) \ for \ all \ (x, t) \in \overline{\Omega}_1 \times [0, 1]. \ Thus

\[ \deg_K(h_0, \Omega) = \deg_K(h_0, U_1) + \deg_K(h_0, \Omega_1) \]

\[ = \deg_K(f, U_1) + \deg_K(f, \Omega_1) = \deg_K(f, \Omega). \]

Since \ h^{(2)} \ determines \ a \ G-homotopy \ between \ f \ and \ h_1 \ in \ C_0(\overline{\Omega}, \overline{\Omega} \setminus (\Omega_0 \cup U)) \ the \ inductive \ hypothesis \ implies \ \deg_K(f, U) = \deg_K(h_1, U), \ hence

\[ \deg_K(f, \Omega) = \deg_K(f, \Omega_0) + \deg_K(f, U) \]

\[ = \deg_K(h_1, \Omega_0) + \deg_K(h_1, U) = \deg_K(h_1, \Omega). \]

Therefore \ \deg_K(h_0, \Omega) = \deg_K(h_1, \Omega) \ \ and \ \ the \ proof \ of \ 2.13 \ is \ complete.

Proof \ of \ Proposition \ 2.12. \ Suppose \ h \ is \ a \ G-homotopy \ between \ two \ elementary \ G-maps \ f_0, f_1 \in C_0(\overline{\Omega}, \partial \Omega). \ Let \ X = \Pi(h^{-1}(0)), \ where \ \Pi \ denotes \ the \ projection \ of \ \overline{\Omega} \times [0, 1] \ onto \ \overline{\Omega}. \ Clearly \ X \ is \ a \ compact \ subset \ of \ \Omega. \ There \ exists \ a \ family \ of \ slice \ maps \ \phi_i: B^n(2) \to \Omega, \ i = 1, \ldots, r, \ such \ that \ \{G \phi_i(B^n)\} \ covers \ X. \ Let \ U_i = G \phi_i(B^n). \ By \ 2.10, \ \overline{\Omega}, \partial \Omega \cup \partial U_1 \cup \ldots \cup \partial U_r) \ is \ open \ and \ dense \ in \ \overline{\Omega}, \partial \Omega). \ By \ 2.8 \ there \ exist \ F_0, F_1 \in C_0(\overline{\Omega}, \partial \Omega \cup \partial U_1 \cup \ldots \cup \partial U_r) \ such \ that

\[(1-t)f_0 + tf_1, (1-t)f_1 + tf_1 \in C_0(\overline{\Omega}, (U_1 \cup \ldots \cup U_r)) \]

and \ are \ elementary \ for \ all \ t \in [0, 1]. \ Let
$$H(x, t) = \begin{cases} 3t \cdot f_0(x) + (1 - 3t) \cdot F_0(x) & \text{for } 0 \leq t \leq \frac{1}{3}, \\ h(x, 3t - 1) & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}, \\ (3 - 3t) \cdot f_1(x) + (3t - 2) \cdot F_1(x) & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases}$$

Since $H$ satisfies the assumptions of 2.13 we have $\deg_K(F_0, \Omega) = \deg_K(F_1, \Omega)$. On the other hand, by 2.9, $\deg_K(f_0, \Omega) = \deg_K(F_0, \Omega)$ and $\deg_K(f_1, \Omega) = \deg_K(F_1, \Omega)$. Therefore $\deg_K(f_0, \Omega) = \deg_K(f_1, \Omega)$ and the proof is complete.

Now, our goal is to extend the definition of $\deg_K(f, \Omega)$ from elementary $G$-maps to all $f \in C_0(\bar{\Omega}, \bar{\partial}\Omega)$. To begin with, let $f \in C_0(\bar{\Omega}, \bar{\partial}\Omega)$ and $\varepsilon = \inf \{|f(x)| : x \in \partial\Omega\}$. By 2.11 there exists an elementary $G$-map $f_0 \in C_0(\bar{\Omega}, \bar{\partial}\Omega)$ such that $\|f - f_0\| < \varepsilon$. We define

$$\deg_K(f, \Omega) = \deg_K(f_0, \Omega).$$

To see that our definition is independent of the choice of $f_0$ choose another elementary $G$-map $f_1$ such that $\|f - f_1\| < \varepsilon$. Then, by 2.5, both $f_0$ and $f_1$ are homotopic to $f$ in $C_0(\bar{\Omega}, \bar{\partial}\Omega)$. Therefore $f_0$ and $f_1$ are two elementary $G$-maps which are homotopic in $C_0(\bar{\Omega}, \bar{\partial}\Omega)$ and by 2.12, $\deg_K(f_1, \Omega) = \deg_K(f_0, \Omega)$. From the definition of $\deg_K(f, \Omega)$ and Properties (I)-(VI) of the classical degree we obtain the following result.

**2.14. Theorem.** Suppose that $V$ is a representation of $S^1$. Let $K$ denote the smallest isotropy group of $V$ and assume $K \neq S^1$. Let $\Omega$ run through the family of all open bounded invariant subsets of $V \oplus \mathbb{R}$ such that $\Omega_K = \Omega$, and $f : X \to V$ through $G$-maps such that $X$ is invariant, $\Omega \subset X$ and $f(\partial\Omega) \subset V \setminus \{0\}$. Then there exists a $Z$-valued function $\deg_K(f, \Omega)$ satisfying the following conditions:

(a) If $\deg_K(f, \Omega) \neq 0$ then there exists $x \in \Omega$ such that $f(x) = 0$.
(b) If $\Omega_0 \subset \Omega$ is open, invariant and $f^{-1}(0) \cap \Omega = \Omega_0$ then $\deg_K(f, \Omega_0) = \deg_K(f, \Omega)$.
(c) If $\Omega_1 \cap \Omega_2$ are open invariant subsets of $\Omega$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $f^{-1}(0) \cap \Omega_1 \cup \Omega_2$ then $\deg_K(f, \Omega) = \deg_K(f, \Omega_1) + \deg_K(f, \Omega_2)$.
(d) If $h : (\Omega \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V \setminus \{0\})$ is a $G$-homotopy then $\deg_K(h_0, \Omega) = \deg_K(h_1, \Omega)$.
(e) Suppose $W$ is another representation of $S^1$ such that $K$ is the smallest isotropy group of $V \oplus W$ and let $U$ be an open bounded invariant subset of $W$ such that $0 \in U$. Define $F : U \times \Omega \to W \oplus V$ by $F(x, y) = (x, f(y))$. Then $\deg_K(F, U \times \Omega) = \deg_K(f, \Omega)$.
(f) Suppose $f$ is of class $C^1$ on $\Omega$, 0 is a regular value of $f$ and $f^{-1}(0) = G\cdot a$. Let $v = \eta'(0)$, where $\eta(t) = g(t) \cdot a$, $g(t) = \exp(2\pi it)$, be the tangent vector to $G\cdot a$ at $a$. Then $\deg_K(f, \Omega) = \text{sgn}(Df(a), v)$.

Finally, we briefly discuss the case $K = S^1$. Thus we assume that the smallest isotropy group equals $G = S^1$, hence $V$ is a trivial representation of $S^1$.
Therefore we let $V = R^n$ and consider a continuous map $f: (\Omega, \partial \Omega) \rightarrow (R^n, R^n \setminus \{0\})$, where $\Omega$ is an open bounded subset of $R^{n+1}$. Let $R^n$ denote the one-point compactification of $R^n$. It can easily be shown that there exists $\tilde{f}: R^{n+1} \rightarrow R^n$ which extends $f$ and $\tilde{f}(R^{n+1} \setminus \Omega) \subset R^n \setminus \{0\}$. Moreover, the homotopy class of $f$ depends only on the homotopy class of $\tilde{f}$. Since $R^n$ is homeomorphic to $S^n$ we may identify $[R^{n+1}, R^n]$ with the $(n+1)$th homotopy group of $S^n$. Recall that the suspension homomorphism determines a homomorphism $\Sigma: \pi_{n+1}(S^n) \rightarrow Z_2$, and that $\Sigma$ is an isomorphism for $n \geq 3$. Define $\deg_\sigma(f, \Omega) = \Sigma([\tilde{f}]) \in Z_2$. The following theorem follows easily from the results proved in Gęba–Massabó–Vignoli [12].

**2.15. Theorem.** Suppose that $V$, $\dim V \geq 1$, is a trivial representation of $G = S^1$. Let $\Omega$ run through the family of all open bounded invariant subsets of $V \oplus R$ and $f: X \rightarrow V$ through continuous maps such that $\Omega \subset X$ and $f(\partial \Omega) \subset V \setminus \{0\}$. Then there exists a $Z_2$-valued function $\deg_\sigma(f, \Omega)$ satisfying the following conditions:

(a) If $\deg_\sigma(f, \Omega) \neq 0$ then there exists $x \in \Omega$ such that $f(x) = 0$.

(b) If $\Omega_0 \subset \Omega$ is open and $f^{-1}(0) \cap \Omega = \Omega_0$ then $\deg_\sigma(f, \Omega_0) = \deg_\sigma(f, \Omega)$.

(c) If $\Omega_1, \Omega_2$ are open subsets of $\Omega$ such that $\Omega_1 \cup \Omega_2 = \Omega$ and $f^{-1}(0) \cap \Omega = \Omega_1 \cup \Omega_2$ then $\deg_\sigma(f, \Omega) = \deg_\sigma(f, \Omega_1) + \deg_\sigma(f, \Omega_2)$.

(d) If $h: (\tilde{\Omega} \times [0, 1], \partial \Omega \times [0, 1]) \rightarrow (V, V \setminus \{0\})$ is a homotopy then $\deg_\sigma(h, \Omega) = \deg_\sigma(h_0, \Omega)$.

(e) Suppose $W$ is another trivial representation of $S^1$ and let $U$ be an open bounded subset of $W$ such that $0 \in U$. Define $F: U \times \tilde{\Omega} \rightarrow W \oplus V$ by $F(x, y) = (x, f(y))$. Then $\deg_\sigma(F, U \times \Omega) = \deg_\sigma(f, \Omega)$.

**3. Proof of the general case of Theorem 1.2.** In this section, using the results of the preceding section, we complete the proof of Theorem 1.2. In view of 1.1 we may assume that

\[ V = R[k_0, 0] \oplus R[k_1, m_1] \oplus \cdots \oplus R[k_r, m_r] \]

is a fixed representation of $G = S^1$. In what follows we identify the linear space $V$ with $R^n$, where $n = k_0 + 2k_1 + \cdots + 2k_r$. This determines an orientation of $V$. Moreover, with respect to the standard scalar product in $R^n$, $V$ is an orthogonal representation of $S^1$ $\langle x, y \rangle = \langle gx, gy \rangle$ for all $g \in S^1$ and $x, y \in V$.

Let $H = \{H_1, \ldots, H_k\}$ denote the family of all closed subgroups $S^1$ such that $V_H \neq \emptyset$. If $H \in H$ then $V_H$ (resp. $(V \oplus R)^H$) is a $G$-invariant linear space of $V$ (resp. $V \oplus R$). Note also that $(V \oplus R)^H = V^H \oplus R$ for all $H \in H$. Since $V$ and $V \oplus R$ are orthogonal representations of $S^1$ we have direct sum decompositions $V = V^H \oplus (V^H)^\perp$, $V \oplus R = (V \oplus R)^H \oplus (V^H)^\perp$. Let $P^H: V \oplus R \rightarrow (V^H)^\perp$ denote the orthogonal projection. Note that $(V^H)^\perp$ is a $G$-invariant subspace of $V$.

In this section we still use all the notations of Sections 1 and 2.

Assume now that $\Omega \subset V \oplus R$ is $G$-invariant, open and bounded.
3.1. Definition. We say that a map \( f \in C_G(\Omega, \partial \Omega) \) is normal if there exists \( \eta > 0 \) such that \( |P^H x| < \eta \) implies \( P^H x = P^H f(x) \) for all \( x \in \Omega \) and \( H \in H \). Similarly we say that a G-homotopy \( h: (\Omega \times [0, 1], \partial \Omega \times [0, 1]) \rightarrow (V, V \setminus \{0\}) \) is normal if there exists \( \eta > 0 \) such that \( |P^H x| < \eta \) implies \( P^H h(x, t) = P^H x \) for all \( (x, t) \in \Omega \times [0, 1] \) and \( H \in H \).

The following fact is an immediate consequence of Definition 3.1 and the observation that \( \Omega_H \) is open in \( V^H \).

3.2. Corollary. Suppose \( f \in C_G(\Omega, \partial \Omega) \) is normal. For \( H \in H \) let \( f_H: \Omega_H \rightarrow V^H \) denote the restriction. Then \( f_H^{-1}(0) \) is a compact subset of \( \Omega_H \). Similarly, if \( h: (\Omega \times [0, 1], \partial \Omega \times [0, 1]) \rightarrow (V, V \setminus \{0\}) \) is a normal homotopy and \( h_H: \Omega_H \times [0, 1] \rightarrow V^H \) denotes the restriction then \( h_H^{-1}(0) \) is a compact subset of \( \Omega_H \times [0, 1] \).

3.3. Proposition. Let \( \Omega \) be an open bounded G-invariant subset of \( V \oplus R \).

(*) For any \( f \in C_G(\Omega, \partial \Omega) \) there exists \( f_0 \in C_G(\Omega, \partial \Omega) \) such that \( f_0 \) is normal and G-homotopic to \( f \).

(**) If \( f_0, f_1 \in C_G(\Omega, \partial \Omega) \) are two G-homotopic normal maps then there exists a normal homotopy between \( f_0 \) and \( f_1 \).

In the proof of 3.3 we will use the following lemmas.

3.4. Lemma. For every \( \varepsilon > 0 \) and \( H \in H \) there exists a G-equivariant map \( r: V \oplus R \rightarrow V \oplus R \) such that

(*) \( |r(x) - x| < \varepsilon \) for all \( x \in V \oplus R \),

(**) if \( x \in V \oplus R \) and \( |P^H x| \leq \varepsilon \) then \( r(x) \in V^H \oplus R \).

Proof. Define \( \theta: V \oplus R \rightarrow [0, 1] \) by

\[
\theta(x) = \begin{cases} 
0 & \text{if } |P^H(x)| \leq \varepsilon, \\
(|P^H(x)| - \varepsilon) \varepsilon^{-1} & \text{if } \varepsilon < |P^H(x)| < 2\varepsilon, \\
1 & \text{if } |P^H(x)| \geq 2\varepsilon.
\end{cases}
\]

Define \( r \) by \( r(x) = x - P^H(x) + \theta(x)P^H(x) \). It is easy to check that \( r \) has the required properties.

3.5. Lemma. For every \( \delta > 0 \) there exist \( \eta > 0 \) and a G-map \( \Gamma: V \oplus R \rightarrow V \oplus R \) such that

(*) \( |\Gamma(x) - x| \leq \delta \) for all \( x \in V \oplus R \),

(**) if \( H \in H \), \( x \in V \oplus R \) and \( |P^H x| \leq \eta \) then \( \Gamma(x) \in V^H \oplus R \).

Proof. For \( H_i \in H = \{H_1, \ldots, H_k\} \) we set \( P_i = P^{H_i} \). Given \( \delta > 0 \) set \( \delta_i = \varepsilon_i^{-1} \). From the preceding lemma it follows that for each \( i, 1 \leq i \leq k \), there exists \( r_i: V \oplus R \rightarrow V \oplus R \) such that \( |r_i(x) - x| \leq \delta_i \) for all \( x \in V \oplus R \) and \( |P_i(x)| \leq \delta_i \) implies \( r_i(x) \in V^{H_i} \oplus R \). Set \( \Gamma = r_1 \circ \ldots \circ r_k \). An easy computation shows that \( \Gamma \) has all the required properties.
Proof of 3.3. First, to prove (⋆) suppose $f: (\bar{\Omega}, \partial\Omega) \to (V, V \setminus \{0\})$ is a $G$-map. Choose $\varepsilon > 0$ such that $|f(x)| \geq 3\varepsilon$ for all $x \in \partial\Omega$. Let $D = \{x \in V \oplus R: |x| \leq R\}$ be a closed disc such that $\Omega \subset \{x \in V \oplus R: |x| \leq R - \varepsilon\}$. By the Tietze–Gleason extension theorem (Bredon [3]) there exists a $G$-map $F: D \to V$ which extends $f$. Since $F$ is uniformly continuous, there exists $\delta, 0 < \delta \leq \varepsilon$, such that $x, y \in D$ and $|x - y| < \delta$ imply $|F(x) - F(y)| < \varepsilon$. Applying Lemma 3.5 we get $\Gamma: V \oplus R \to V \oplus R$ and $\eta > 0$ which satisfy conditions (⋆) and (⋆⋆) of that lemma. Since $\delta \leq \varepsilon$ and $|\Gamma(x) - x| < \delta$, we have $\Gamma(\bar{\Omega}) \subset D$. Let $Q: V \oplus R \to V$ denote the projection map. For $(x, t) \in \bar{\Omega} \times I$ let

$$h(x, t) = F((1 - t)x + t\Gamma(x)) + tQ(x - \Gamma(x)).$$

If $(x, t) \in \partial\Omega \times I$ then $|x - \Gamma(x)| \leq \delta \leq \varepsilon$, which implies

$$|Q(x - \Gamma(x))| \leq \varepsilon \quad \text{and} \quad |(1 - t)x + t\Gamma(x) - x| = |t(\Gamma(x) - x)| \leq \delta$$

and so $|F((1 - t)x + t\Gamma(x)) - F(x)| \leq \varepsilon$. Thus for $(x, t) \in \partial\Omega \times [0, 1]$ we have

$$|h(x, t)| = |h(x, t) - f(x) + f(x)| \geq |f(x) - |F((1 - t)x + t\Gamma(x)) - F(x)| - |Q(x - \Gamma(x))| \geq \varepsilon.$$

Therefore $h: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V \setminus \{0\})$ is a $G$-homotopy. Let $f_0 = h_1$.

If for some $H \in H$ and $x \in \bar{\Omega}$, $|P^H(x)| \leq \eta$, then $\Gamma(x) \in V^H \oplus R$, hence $F(\Gamma(x)) \in V^H$ and thus $P^H f_0(x) = P^H(F(\Gamma(x)) + x - \Gamma(x)) = P^H(x)$. Therefore $f_0$ is normal and the proof of (⋆) is complete.

To prove (⋆⋆) assume $h: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V \setminus \{0\})$ is a $G$-homotopy such that $f_0 = h_0$ and $f_1 = h_1$ are normal. By the definition of a normal map

(a) there exists $\eta_1 > 0$ such that $P^H(f_i(x)) = P^H(x)$ for all $x \in \bar{\Omega}$ such that $P^H(x) \leq \eta_0$ and $H \in H$ and $i = 0, 1$.

Choose $\varepsilon > 0$ such that $|h(x, t)| \geq 3\varepsilon$ for all $(x, t) \in \partial\Omega \times [0, 1]$. Arguing as in the proof of (⋆) we extend $h$ to a $G$-map $\Phi: D \times [0, 1] \to V$ and choose $\delta > 0$ such that $|x - y| + |t - s| < \delta$ implies $|\Phi(x, t) - \Phi(y, s)| < \varepsilon$. Applying Lemma 3.5 we obtain $\eta > 0$ and $\Gamma$ satisfying conditions (⋆) and (⋆⋆) of that lemma.

For $(x, t) \in \Omega \times [0, 1]$ set $h^*(x, t) = \Phi(\Gamma(x, t) + Q(x - \Gamma(x))$, $h^{(0)}(x, t) = (1 - t)f_0(x) + tf^*(x, 0)$, $h^{(1)}(x, t) = tf_1(x) + (1 - t)h^*(x, 1)$. Evidently

$$h^{(0)}, h^*, h^{(1)}: (\bar{\Omega} \times [0, 1], \partial\Omega \times [0, 1]) \to (V, V \setminus \{0\})$$

are $G$-homotopies. Set $\eta_1 = \min\{\eta_0, \eta\}$. From the definition of $h^*$ it follows at once that $h^*$ is a normal $G$-homotopy. The definition of $h^{(0)}$, $h^{(1)}$ and condition (a) imply $P^H h^{(0)}(x, t) = P^H x$ if $|P^H x| < \eta_1$, $x \in \bar{\Omega}$, $H \in H$, $i = 0, 1$. Therefore $h^{(0)}$, $h^{(1)}$ are normal $G$-homotopies. The composition of the normal homotopies
$h^{(0)}, h^*, h^{(1)}$ defines a normal homotopy between $f_0$ and $f_1$. The proof of 3.3 is complete.

In view of 3.2 the following definition is correct.

3.6. **Definition.** Suppose $f \in C_G(\Omega, \partial \Omega)$ is normal. For each $H \in H$ choose an open invariant subset $U_H$ such that $f_H^{-1}(0) \subset U_H \subset \Omega_H$. Then $\deg_H(f_H, U_H)$ is well defined and independent of the choice of $U_H$. Define

$$
\deg_H(f, \Omega) = \begin{cases} 
\deg_H(f_H, U_H) & \text{for } H \in H, \\
0 & \text{for } H \notin H,
\end{cases}
$$

and $\deg(f, \Omega) = \{\deg_H(f, \Omega)\}$.

Consider now the general case and take $f \in C_G(\Omega, \partial \Omega)$ not necessarily normal. By Corollary 3.2 there exists $f_0 \in C_G(\Omega, \partial \Omega)$ normal and $G$-homotopic to $f$. Moreover, if $f_1 \in C_G(\Omega, \partial \Omega)$ is another normal map homotopic to $f$ then there exists a normal homotopy between $f_0$ and $f_1$. Hence, by 2.14 and 2.15, $\deg(f_0, \Omega) = \deg(f_1, \Omega)$. Therefore the following definition is correct.

3.7. **Definition.** For $f \in C_G(\Omega, \partial \Omega)$ let $\deg(f, \Omega) = \deg(f_0, \Omega)$, where $f_0 \in C_G(\Omega, \partial \Omega)$ is normal and $G$-homotopic to $f$.

**Proof of Theorem 1.2.** To prove (a) suppose that $f \in C_G(\Omega, \partial \Omega)$ and $H \in H$ are such that $f(x) \neq 0$ for all $x \in \Omega_H$. Let $\varepsilon = \inf \{|f(x)| : x \in \Omega_H\}$. The arguments used in the proof of condition (a) of 3.3 show that there exists a normal map $f_0$ which is homotopic in $C_G(\Omega, \partial \Omega)$ to $f$ and $|f(x) - f_0(x)| < \varepsilon$ for all $x \in \Omega_H$. Thus $f_0(x) \neq 0$ for all $x \in \Omega_H$. Therefore $\deg(f, \Omega) = \deg(f_0, \Omega) = 0$ and the proof of (a) is complete.

(b), (c) and (e) are easy consequences of 2.14(b), 2.15(b), 2.14(c), 2.15(c), 2.14(e) and 2.15(e). Property (d) follows from Definition 3.6, 2.14(d), 2.15(d) and condition (**) of 3.3. Thus the proof of Theorem 1.2 is complete.

4. **Computations of the $S^1$-degree.** In this section we prove some formulas which express the $S^1$-degree of equivariant map in terms of its derivative.

We keep the notations of the preceding sections. As before we assume that $\Omega$ is an open invariant bounded subset of $V \oplus R$, where $V$ is an orthogonal representation of $G = S^1$ of the form described by condition (a) given at the beginning of Section 3. Suppose $f \in C_G(\Omega, \partial \Omega)$ and let $f_0 : \Omega \to V$ denote the restriction of $f$. We assume further that $f_0$ is of class $C^1$ and $0$ is a regular value of $f_0$. In this case $f^{-1}(0) = M_1 \cup \ldots \cup M_k$, where each $M_i$ is diffeomorphic to $S^1$ and $M_i \cap M_j = \emptyset$ for $i \neq j$. For each $i$ choose an open invariant subset $\Omega_i \subset \Omega$ such that $M_i \subset \Omega_i$ and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. Clearly $\deg(f, \Omega) = \deg(f, \Omega_1) + \ldots + \deg(f, \Omega_k)$. Therefore throughout the rest of this section we assume $k = 1$, i.e. we assume $f^{-1}(0) = M$ is connected, hence diffeomorphic to $S^1$. We also let $K$ denote the isotropy group of points in $M$. We discuss separately the following two cases:
Case 1. $K \neq S^1$, $K = Z_m$.
Case 2. $K = S^1$.

Having in mind our applications of the equivariant degree to differential equations, we first discuss Case 1. Choose $a \in M$ and let $A = Df(a)$. Differentiating, for $g \in K$, the equality $f(ga) = gf(a)$ we see that $A: V \oplus R \to V$ is a $K$-equivariant linear map. By the Schur Lemma, the direct sum decompositions

$$V \oplus R = (V^K \oplus R) \oplus (V^K)\perp, \quad V = V^K \oplus (V^K)\perp$$

give a decomposition of $A$ of the form

$$A = \begin{bmatrix} A^K & 0 \\ 0 & A^\perp \end{bmatrix},$$

where $A^K: V^K \oplus R \to V^K$, $A^\perp: (V^K)\perp \to (V^K)\perp$.

Since $M = Ga$ is diffeomorphic to $S^1/K$, it is an oriented 1-dimensional submanifold of $\Omega$. Set $v = \eta'(0)$, where $\eta(t) = g(t)a$, $g(t) = \exp(2\pi it)$. Clearly $v \in V^K \oplus R$. Let $N^K = \{x \in V^K \oplus R : \langle x, v \rangle = 0\}$, i.e. $N^K$ is the linear subspace of $V^K \oplus R$ which is normal to $M$ at $a$.

The following theorem describes the $S^1$-degree of the map $f$ in question.

**4.1. Theorem.** Suppose $f \in C_G(\overline{\Omega}, \partial\Omega)$ satisfies all the assumptions of Case 1.

(i) If $\det A^\perp > 0$ then

$$\deg_H(f, \Omega) = \begin{cases} \text{sgn}(A^K, v) & \text{if } H = K = Z_m, \\
0 & \text{if } H \neq K. \end{cases}$$

(ii) If $\det A^\perp < 0$ then $m = 2\mu$ is even and

$$\deg_H(f, \Omega) = \begin{cases} \text{sgn}(A^K, v) & \text{if } H = K = Z_m, \\
-\text{sgn}(A^K, v) & \text{if } H = Z_\mu, \\
0 & \text{if } H \neq Z_m, Z_\mu. \end{cases}$$

In the proof of Theorem 4.1 we will need the following two facts. The first is essentially a version of the Slice Theorem (see Bredon [3]).

**4.2. Proposition.** Assume $\Omega = GB$, where $B$ is an open sufficiently small disc in the space $N = \{x \in V \oplus R : \langle x, v \rangle = 0\}$ with center $a$. Then every $K$-equivariant map $\varphi \in C_K(B, \partial B)$ (every $K$-equivariant homotopy $\chi \in C_K(B \times \{0, 1\}, \partial B \times \{0, 1\})$) determines a $G$-equivariant map $f \in C_G(\overline{\Omega}, \partial\Omega)$ ($G$-equivariant homotopy $h \in C_G(\overline{\Omega} \times \{0, 1\}, \partial\Omega \times \{0, 1\})$) by

$$f(g \cdot x) = g \cdot \varphi(x) \quad (h(g \cdot x, t) = g \cdot \chi(x, t)),$$

where $x \in B$, $g \in G = S^1$.

**4.3. Lemma.** Suppose that $W$ is a representation of $K = Z_m$ such that $W^K = \{0\}$ and let $\text{Aut}_K W$ denote the group of all $K$-equivariant linear automorphisms of $W$. 

(*) If $A \in \text{Aut}_K(W)$ and $\det A > 0$ then $A$ belongs to the connected component of $\text{Aut}_K W$ containing the identity.

(**) If $A \in \text{Aut}_K W$ and $\det A < 0$ then there exist subrepresentations $W_1, W_2$ of $W$ such that $W = W_1 \oplus W_2$, $W_1$ is one-dimensional and $A$ belongs to the connected component of $J$, where (with respect to the above decomposition)

$$J = \begin{bmatrix}
-I & 0 \\
0 & I
\end{bmatrix}.$$

The above lemma is a consequence of some standard facts in group representation theory. For the sake of completeness we will sketch the proof of it after proving Theorem 4.1.

Proof of Theorem 4.1. Notice that without loss of generality we may assume that $\Omega = GB$, where $B$ denotes a sufficiently small disc in $N$ with center at $a$. Recall that $A = Df(a)$ is a $K$-equivariant linear map and $A_{in}: N \to V$ is a $K$-equivariant isomorphism. Hence the map $\varphi: B \to V$ defined by $\varphi(x) = A(x-a)$ belongs to $C_K(B, \partial B)$. If the disc $B$ is sufficiently small then the linear homotopy $\chi(x, t) = tf_{in}(x) + (1-t)\varphi(x)$ is zero only for $x = a$. Applying Lemma 4.3 to $A^1 \in \text{Aut}_K((V^K)^1)$ we obtain a continuous path $\eta: [0, 1] \to \text{Aut}_K((V^K)^1)$ such that $\eta(0) = A^1$ and $\eta(1) = A_1$, where $A_1$ is either $I$ or $J$.

Define $\varphi^* \in C_K(B, \partial B)$ by $\varphi^*(x) = A_*(x-a)$, where, with respect to the direct sum decompositions $V \oplus R = (V^K \oplus R) \oplus (V^K)^1$ and $V = V^K \oplus (V^K)^1$,

$$A_* = \begin{bmatrix}
A^K & 0 \\
0 & A_1
\end{bmatrix}.$$

Define $f_1 \in C_G(\Omega, \partial \Omega)$ by $f_1(\theta x) = \theta \varphi^*(x)$, where $x \in B$, $\theta \in G = S^1$. Proposition 4.2 implies that $f_1$ is homotopic in $C_G(\Omega, \partial \Omega)$ to $f$. If $\det A^1 > 0$ then the first part of our theorem follows easily from 1.2(e).

If $\det A^1 < 0$ then by 4.3 there exists a direct sum decomposition $W = (V^K)^1 = W_1 \oplus W_2$ such that $W_1$ is one-dimensional and $A_1 = J$. Let $\Omega_0 = \Omega \cap (V^K \oplus R)$ and, for $i = 1, 2$, let $U_i$ denote the unit disc in $W_i$. Without loss of generality we may assume $\Omega = \Omega_0 \times U_1 \times U_2$. Let $F: \Omega_0 \times U_1 \to V^K \oplus W_1$ denote the restriction of $f_1$. Writing points of $(V^K \oplus R) \oplus W_1$ in the form $x = (y, z)$, we have $F(y, z) = (f_1(y), -z)$. Let $\psi: W_1 \to W_1$ be an odd $C^1$-function such that $\psi(z) = z$ for $||z|| \leq \frac{1}{2}$, $\psi(z) = -z$ for $||z|| \geq \frac{1}{2}$, $\psi^{-1}(0) = \{-\tau, 0, \tau\}$ and $\psi'(\tau) = -1$. The formula $\psi(z) = (1-t)\psi(z) + t(-z)$ gives a homotopy between $-1: W_1 \to W_1$ and $\psi$. Define $F_1: \Omega_0 \times U_1 \to V^K \oplus W_1$ by $F_1(y, z) = (f_1(y), \psi(z))$. From the definition of $F_1$ and 4.2 it follows at once that $F_1 \in C_G(\Omega_0 \times U_1, \partial(\Omega_0 \times U_1))$ and $F_1$ is homotopic to $F$ in $C_G(\Omega_0 \times U_1, \partial(\Omega_0 \times U_1))$. Obviously $(F_1)^{-1}(0) = M \cup M_1$, where $M_1 = Gb$ and $b = (a, \tau) \in \Omega_0 \times U_1$ has isotropy group $Z_a$. Applying the first part of our theorem to $F_1$ we obtain
\[ \deg_H(F, \Omega_0 \times U_1) = \begin{cases} 
\text{sgn}(A^K, v) & \text{if } H = K = Z_m, \\
-\text{sgn}(A^K, v) & \text{if } H = Z_\mu, \\
0 & \text{if } H \neq Z_m, Z_\mu. 
\end{cases} \]

Applying again 1.2(e) we have \( \deg_H(f, \Omega) = \deg_H(F, \Omega_0 \times U_1) \) for all \( H \in H \) and the proof of Theorem 4.1 is complete.

To sketch the proof of Lemma 4.3 we have to collect some basic facts concerning representations of the group \( K = Z_m \). In what follows we identify \( Z_m \) with a subgroup of \( S^1 \). Let \( \gamma \) denote the generator of \( Z_m \) corresponding to \( \exp(2\pi i/m) \). Suppose \( W \) is a representation of \( Z_m \). According to the conventions we have adopted in Section 1, \( W = (W, \varrho) \), where \( W \) is a finite-dimensional linear space over \( R \) and \( \varrho: Z_m \rightarrow GL(W) \) is a group homomorphism. \( W \) is called irreducible if there is no decomposition \( W = W_1 \oplus W_2 \) such that both \( W_1 \) and \( W_2 \) are positive-dimensional subrepresentations of \( W \).

Let \( L_K(W) \) denote the linear space of all linear \( K \)-equivariant maps of \( W \) into itself and \( \text{Aut}_K(W) \) the group of all linear \( K \)-equivariant automorphisms of \( W \): \( I \in \text{Aut}_K(W) \) will denote the identity. For \( j \in \mathbb{N} \) define \( \varrho_j: Z_m \rightarrow GL(2, R) \) by

\[ \varrho_j(e^{i\theta}) = \begin{bmatrix} \cos j\theta & -\sin j\theta \\
\sin j\theta & \cos j\theta \end{bmatrix} \]

Let \( Q_j = (R^2, \varrho_j) \). We will need the following fact which is an easy consequence of the introduced definitions.

**4.4. Lemma.** For given \( j \in \mathbb{N} \) let \( A_1 = \varrho_j(\gamma) \). If \( A_1 \neq I, -I \) then \( L_K(Q_j) \) and \( C \) are isomorphic as linear spaces over \( R \).

As a direct consequence of 4.4 we obtain the following simple observation.

**4.5. Corollary.** Suppose \( L_K(Q_j) \simeq C \) and \( W \) is the direct sum of \( k \) copies of \( Q_j \). Then \( \text{Aut}_K(W) \) is isomorphic to \( GL(k, C) \) (= the group of all \( C \)-linear automorphisms of \( C^k \)).

In the case where \( m \) is even and \( m = 2\mu \) we let \( Q_0 = (R, \varrho_0) \) where \( \varrho_0: Z_m \rightarrow GL(1, R) \) is defined by \( \varrho_0(\gamma) = -I \). In this case every linear map of \( Q_0 \) into itself is \( K \)-equivariant. Therefore we have the following:

**4.6. Corollary.** Suppose \( W \) is the direct sum of \( k \) copies of \( Q_0 \). Then \( \text{Aut}_K(W) \) is isomorphic to \( GL(k, R) \).

**4.7. Proposition** (Serre [20]). Suppose \( W \) is a nontrivial irreducible representation of \( Z_m \).

(*) If \( m \) is odd and \( \mu = (m-1)/2 \) then \( W \) is 2-dimensional and there exists \( j \), \( 1 \leq j \leq \mu \), such that \( W \) is equivalent to \( Q_j \). Moreover, if \( 1 \leq \alpha < \beta \leq \mu \) then \( Q_\alpha \) is not equivalent to \( Q_\beta \).
(**) If $m$ is even and $\mu = m/2$ then either $W$ is 2-dimensional and there exists $j$, $1 \leq j < \mu$, such that $W$ is equivalent to $Q_j$, or $W$ is 1-dimensional and equivalent to $Q_0$. Moreover, if $0 \leq \alpha < \beta \leq \mu$ then $Q_\alpha$ is not equivalent to $Q_\beta$.

4.8. Proposition. Suppose that $W$ is a representation of $K = Z_m$ such that $W^K = \{0\}$. Then admitting $W_j = 0$ as an empty sum, we have:

(*) If $m$ is odd then $W = W_1 \oplus \ldots \oplus W_\mu$, where $\mu = (m-1)/2$ and each $W_j$ is equivalent to direct sum of $Q_j$.

(**) If $m$ is even and $m = 2\mu$ then $W = W_0 \oplus W_1 \oplus \ldots \oplus W_{\mu-1}$, where each $W_j$ is equivalent to a direct sum of $Q_j$.

Proof of Lemma 4.3. The Schur Lemma implies that with respect to the direct sum decomposition $W = W_0 \oplus W_1 \oplus \ldots \oplus W_\mu$ the linear map $A \in \text{Aut}_K W$ is of the diagonal form

$$A = \begin{bmatrix}
A_0 & & \\
& A_1 & \\
& & \ddots \\
& & & A_\mu
\end{bmatrix}, \quad \text{where } A_j : W_j \to W_j.$$

Thus $\text{Aut}_K(W) = \text{Aut}_K(W_0) \times \text{Aut}_K(W_1) \times \ldots \times \text{Aut}_K(W_\mu)$. Since $\text{GL}(k, C)$ is connected and $\text{GL}(k, R)$ has two connected components the result follows.

Case 2. $K = G = S^1$.

Recall that in Case 2 we assume that the restriction of $f$ to $\Omega$ is of class $C^1$, $0$ is a regular value of this restriction, $M = f^{-1}(0)$ is connected (therefore diffeomorphic to $S^1$) and $G_x = S^1$ for all $x \in M$. Recall that there is a unique decomposition $V = V^G \oplus V^1 \oplus \ldots \oplus V^k$, where $V^i = R[k_i, m_i]$. For $x \in M$ let $A(x) = Df(x)$, thus the assignment $x \mapsto A(x)$ defines a continuous mapping from $M$ into $L(V \oplus R, V)$. Moreover, for each $x \in M$, $A(x)$ is equivariant and epimorphic. The Schur Lemma implies that, with respect to the direct sum decompositions $V = V^G \oplus V^1 \oplus \ldots \oplus V^k$, $V \oplus R = (V^G \oplus R) \oplus V^1 \oplus \ldots \oplus V^k$, $A(x)$ is of the diagonal form

$$A(x) = \begin{bmatrix}
A^G(x) & 0 \\
& A^1(x) \\
& & \ddots \\
& & & A^k(x)
\end{bmatrix},$$

where $A^G(x) : V^G \oplus R \to V^G$ is a linear epimorphism and $A_i(x) : V^i \to V^i$ are linear isomorphisms.

The $S^1$ action defines a complex structure on $V^j$ by $i \ast x = gx$, where $g \in S^1$, $g = e^{i(m/2\mu)}$. Moreover, an $R$-linear map of $V^j$ is equivariant iff it is $C$-linear with respect to this structure. So, for $i = 1, \ldots, k$, $V^i$ are complex linear spaces and $A_i(x)$ are complex automorphisms (cf. 4.5).

Recall that $\pi_1(\text{GL}(m, C))$ (the fundamental group of $\text{GL}(m, C))$ is isomorphic to $Z$ and the isomorphism is induced by det. Let $d : S^1 \to M$ be a fixed
diffeomorphism such that \( \text{sgn}(A^G(x), v) = -1 \) where \( v = d'(z)(1) \), \( z = d^{-1}(x) \).

We set \( \gamma_i = A_i \circ d: S^1 \to GL(V^i, C) \). Let \( p_i \in Z \approx \pi_1(GL(V^i, C)) \) denote the class of the loop \( \gamma_i \).

4.9. Theorem. Suppose \( f \in C_0(\Omega, \partial\Omega) \) satisfies all the assumptions of Case 2. Then

\[
\text{deg}_H(f, \Omega) = \begin{cases} 
p_0 & \text{if } H = S^1 = G, \\
p_1 & \text{if } H = Z_m, \\
0 & \text{if } H \neq Z_m \text{ and } H \neq G,
\end{cases}
\]

where \( p_0 \in Z_2 \) is defined in Section 2.

The idea of the proof is to find a normal \( G \)-map of simplest form in the homotopy class of \( f \). We begin with the following statement whose proof is left to the reader.

4.10. Lemma. Suppose that \( V \) is a complex linear space of dimension \( k \) and \( e_1, \ldots, e_k \) is a basis of \( V \) over \( C \). Then for every map \( \gamma: S^1 \to GL(V, C) \) there exists a homotopy \( \Lambda: S^1 \times I \to GL(V, C) \) such that \( \Lambda(z, 0) = \gamma(z) \) and

\[
\Lambda(z, 1) = \begin{bmatrix} z^p \\
1 \\
1 \end{bmatrix}
\]

in the basis \( e_1, \ldots, e_k \),

where \( p \) is the class of \( \gamma \) in \( \pi_1(GL(V, C)) \).

Denote by \( U_0 \) a tubular neighbourhood of \( M \) in the open set \( \Omega^G \subset V^G \oplus R \). Identifying \( U_0 \) with a closed neighbourhood of the zero section in the normal bundle, we define a retraction \( r: U_0 \to M \) by

\[
r(x) = x_0 \quad \text{if} \quad x \in x_0 + N_{x_0}.
\]

Next, let \( D_1, \ldots, D_k \) be discs in \( V^1, \ldots, V^k \) respectively such that \( \Omega_1 = U_0 \times D_1 \times \cdots \times D_k \) is contained in \( \Omega \). For \( \gamma_j = A_j \circ d: S^1 \to GL(V^j, C) \) we denote by \( A_j \) the map \( A_j(d^{-1}(\cdot), 1): M \to GL(V^j, C) \), where \( A_j \) is given by Lemma 4.10 for \( \gamma = \gamma_j \).

4.11. Proposition. Let \( f \) satisfy the hypothesis of Theorem 4.9. Then for sufficiently small \( U_0, D_1, \ldots, D_k \) the \( G \)-map \( f_i: (\Omega_1, \partial\Omega_1) \to (V, V \setminus \{0\}) \) given by

\[
f_i(x_1, y_1, \ldots, y_k) = A^G(r(x))(x - r(x)) + \sum A_j(r(x))y_j
\]

is \( G \)-homotopic to \( f \) in \( C_0(\Omega_1, \partial\Omega_1) \).

Proof. Since \( A_j(x): V^j \to V^j \), \( 1 \leq j \leq k \), are \( G \)-equivariant, \( f_i \) is \( G \)-equivariant. It is evident that the \( G \)-map \( \mathcal{H}: (\Omega_1 \times I, \partial\Omega_1 \times I) \to (V, V \setminus \{0\}) \) given by

\[
\mathcal{H}(x, y_1, \ldots, y_k, t) = A^G(r(x))(x - r(x)) + \sum A_j(d^{-1}r(x), t)y_j
\]

yields a \( G \)-homotopy between \( f_i \) and the map

\[
A^G(r(x))(x - r(x)) + \sum A_j(r(x))y_j.
\]
This last map is $G$-homotopic to $f$ in $C_{g}(\Omega_{1}, \partial\Omega_{1})$ by a Rouché principle argument. This proves the proposition.

In the remainder of the proof of Theorem 4.9 we assume $\Omega_{1} = \Omega = U_{0} \times D_{1} \times \cdots \times D_{k}$ to be as in Proposition 4.11. Starting from the map $f_{1}$ we next construct a normal $G$-map $f_{2}$ $G$-homotopic to $f_{1}$. For this purpose we introduce maps $h_{j}: M \times D_{j} \rightarrow V_{j}$, $1 \leq j \leq k$, setting

$$h_{j}(x, y) = \begin{cases} y & \text{if } ||y|| \leq \frac{1}{2} \varepsilon, \\ \alpha A_{j}(x)(6||y||)(1 - \alpha)(\varepsilon/2||y||)y & \text{if } \frac{1}{2} \varepsilon \leq ||y|| \leq \varepsilon, \end{cases}$$

where $\varepsilon$ is the radius of $D_{j}$ and $\alpha = 2||y||/\varepsilon - 1$.

**4.12. Lemma.** Let $h_{j}$ be as defined above.

1. $h_{j}$ is $G$-equivariant as a map of $y$.
2. If $||y|| = \varepsilon$ then $h_{j}(x, y) = A_{j}(x)y$.
3. If $h_{j}(x, y) = 0$ for $y \neq 0$ then

$$A_{j}(x) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

(The last means that $(d^{-1}(x))^{p_{j}} = -1$, or equivalently, $x$ lies in the image of the set $C_{j} = \{ z \in S^{1}: z^{p_{j}} = -1 \}$.)

4. For every $x \in C_{j}$ there exists only one orbit $Gy^{0} \subset D_{j}$ such that $h_{j}(x, y) = 0$ for any $y \in Gy^{0}$. In fact, $y^{0} = (y_{1}, 0, \ldots, 0)$ with $||y^{0}|| = 2s/3$, in the basis $e_{1}, \ldots, e_{k_{j}}$ of $V_{j}$ over $C$ (cf. 4.10).

5. Let $y_{1} = (0, 2s/3)$ in the basis $e_{1}, ie_{1}$ of the plane $C \times \{0\} \times \cdots \times \{0\}$ $\subset V_{j}$, if $w = -e_{1}$ and $n = ie_{1}$, are respectively the tangent and normal vector to $Gy_{1} \subset C$ then the matrix $(\partial h_{j}/\partial y)(x^{0}, y^{0})$, $x^{0} \in d(C_{j})$, from the basis $\{ n, w, e_{2}, ie_{2}, \ldots, e_{k_{j}}, ie_{k_{j}} \}$ to $\{ e_{1}, ie_{1}, e_{2}, ie_{2}, \ldots, e_{k_{j}}, ie_{k_{j}} \}$ has the form

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 \end{bmatrix}$$

6. In the $R$-basis $\{ e_{1}, ie_{1}, e_{2}, ie_{2}, \ldots, e_{k_{j}}, ie_{k_{j}} \}$ we have

$$\frac{\partial h_{j}/\partial y}(x^{0}, y^{0}) = \begin{bmatrix} p_{j}s/3 \\ 0 \\ 0 \end{bmatrix}$$

where $v = d'(z)(1)$, $z = d^{-1}(x^{0})$. 

Proof. An easy computation shows (1)–(4). To prove (5), observe that
\[ h_j(x^0, y^0 + i\varepsilon_1) = h_j(x^0, (2\varepsilon/3 + t)i\varepsilon_1) = [2\varepsilon - 3(2\varepsilon/3 + t)]i\varepsilon_1, \]
which shows that \((\partial h_j/\partial n)(x^0, y^0) = -3i\varepsilon_1\). The second column of the matrix in question is the zero vector, because \(w\) is a tangent vector to \(Gy^0\) and \(h_j\) vanishes on this orbit. Finally, observe that \(h_j\) has the form \(h_j(x, y) = (h_j^*(x, y), y_2, \ldots, y_k)\), which gives (5). Also a direct computation shows (6). This completes the proof.

4.13. Proposition. Let \(h_j(x, y), 1 \leq j \leq k\), be the \(G\)-equivariant map of Lemma 4.12. Then the map \(f_2: (\overline{\Omega}, \partial \Omega) \to (V, V \setminus \{0\})\) given by
\[ f_2(x, y_1, \ldots, y_k) = A^0(r(x))(x - r(x)) + \sum h_j(r(x), y_j) \]
is \(G\)-equivariant, normal and \(G\)-homotopic to \(f_1\) in \(C_0(\overline{\Omega}, \partial \Omega)\).

Proof. Since the \(h_j\) are \(G\)-equivariant, \(f_2\) is \(G\)-equivariant too. We will show that \(f_2\) is normal for the constant \(\varepsilon/2\). For any subgroup \(H \subset S^1\) with \(V_H \neq \emptyset\) we have
\[ (V^H \oplus R)^j = V^j \oplus \ldots \oplus V^j \text{ for some } 1 \leq j_1 < \ldots < j_r \leq k. \]
It follows that \(P^H(x, y_1, \ldots, y_k) = (y_{j_1}, \ldots, y_{j_r})\) for every \((x, y_1, \ldots, y_k) \in \Omega\). If \(\|P^H(x, y_1, \ldots, y_k)\| \leq \varepsilon/2\) then \(\|y_{j_i}\| \leq \varepsilon/2\), and from the definition of \(h_j\), we have \(h_j(r(x), y_j) = y_j\) for \(j = j_1, \ldots, j_r\). It follows that
\[ P^H f_2(x, y_1, \ldots, y_k) = h_{j_1}(r(x), y_{j_1}) + \cdots + h_{j_r}(r(x), y_{j_r}) = y_{j_1} + \cdots + y_{j_r} \]
as desired. Let us join the maps \(f_1\) and \(f_2\) by the linear homotopy \(H((x, y), t) = tf_2(x, y) + (1 - t)f_1(x, y)\). It is sufficient to show that \(H\) has no zero on \(\partial \Omega \times I\). If \((x, y_1, \ldots, y_k) \in \Omega\) then \(x \in \partial V_0\) or \(y_j \in \partial D_1\) for some \(1 \leq j \leq k\). By symmetry, we can assume that \(y_j \in \partial D_1\). Denoting by \(P_1\) the projection onto \(V^1\), we have \(P_1 H((x, y_1, \ldots, y_k, t) = (xH_1(r(x), y_1) + (1 - t)x + A_1(r(x))_1 = A_1(r(x))_1 \neq 0\). Hence \(H((x, y_1, \ldots, y_k, t) \neq 0\). The proof is complete.

Observe that for a given \((y_1, \ldots, y_k)\), if \(f_2(x, y_1, \ldots, y_k) = 0\) then \(x \in \bigcap C_j\) where the \(j\) are such that \(y_j \neq 0\). This means that there exists \((x, y_1, \ldots, y_k) \in \bigcap D^{-1}(0)\) with at least two nonzero coordinates \(y_j, y_j\), if \(C_i \cap C_j \neq \emptyset\). We shall avoid this possibility in the next step of our procedure. For fixed \(C_1, \ldots, C_k\) we can find \(z_1, \ldots, z_k \in S^1\) such that the sets \(z_1 C_1, \ldots, z_k C_k\) are mutually disjoint. Define \(\theta_j: M \to M\) by \(\theta_j(x) = d(z_j^{-1}(d^{-1}(x)))\).

4.14. Proposition. Let \(\theta_j, 1 \leq j \leq k\), be as defined above. Then the map \(f_3: (\Omega, \partial \Omega) \to (V, V \setminus \{0\})\) given by
\[ f_3(x, y_1, \ldots, y_k) = A^0(r(x))(x - r(x)) + \sum h_j(\theta_j r(x), y_j) \]
has the following properties:
(i) $f_3$ is a $G$-equivariant, normal and $G$-homotopic to $f_2$.

(ii) If $f_3(x, y_1, \ldots, y_k) = 0$ then at most one $y_i$ is different from zero. Moreover, $\Omega_H \cap f_3^{-1}(0) = \emptyset$ if $H \neq Z_{m_j}$, $1 \leq j \leq k$, $H \neq S^1$, and $\Omega_H \cap f_3^{-1}(0)$ consists of $p_j$ distinct orbits if $H = Z_{m_j}$.

Proof. Since the $h_j$ are $G$-equivariant in the second variable, $f_3$ is $G$-equivariant. Observe next that homotopies joining the maps $\theta_j$ to the identity on $M$ induce a $G$-homotopy between $f_2$ and $f_3$. By the same argument as in Proposition 4.13, $f_3$ is normal. If $f_3(x, y_1, \ldots, y_k) = 0$ and $y_j \neq 0$ then $\theta_j(x) \in d(C_j)$, or equivalently $d^{-1}(x) \in z_j \cdot C_j$. Since $z_1 C_1, \ldots, z_k C_k$ are mutually disjoint, it follows that only one $y_i$ is different from zero. Suppose that $(x, y_1, \ldots, y_k) \in \Omega_H \cap f_3^{-1}(0)$ where $H = Z_{m_j}$. On account of the above, $(x, y_1, \ldots, y_k) = (x, 0, 0, y_1, 0, \ldots, 0)$. This shows that $\Omega_H \cap f_3^{-1}(0) = \{d(z_j C_j + G y_j)\}$, which means that the last set consists of $p_j$ distinct orbits. Furthermore, if $H \neq Z_{m_j}$, $j = 1, \ldots, k$, $H \neq S^1$ and $\Omega_H \neq \emptyset$ then any point of $\Omega_H$ has at least two coordinates $y_i$ different from zero, which implies $\Omega_H \cap f_3^{-1}(0) = \emptyset$. This ends the proof.

Proof of Theorem 4.9. By $G$-homotopy invariance, $\deg_H(f, \Omega) = \deg_H(f_3, \Omega)$ for every $H$. Suppose that $H \neq S^1$ and $H \neq Z_{m_j}$, $j = 1, \ldots, k$. Since $\Omega_H \cap f_3^{-1}(0) = \emptyset$ and $f_3$ is normal, $\deg_H(f, \Omega) = 0$. Finally, let $H = Z_{m_j}$, and $p = |p_j|$. From what has already been proved, we have

$$\Omega_H \cap f_3^{-1}(0) = \{x_1 + G y, \ldots, x_p + G y\},$$

where $\{x_1, \ldots, x_p\} = d(z_j C_j) \subset M$ and $y \in D_j$, $\frac{1}{2} \varepsilon < ||y|| < \varepsilon$. Fix $x \in \{x_1, \ldots, x_p\}$. The derivative $Df_3(x + y)$ has the form

$$\begin{bmatrix}
A_N & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
D & \cdots & \cdots & \cdots \\
\vdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & 0
\end{bmatrix}$$

with zeros in the remaining places, where $T_x, N_x \subset V^G \oplus \mathbb{R}$ are respectively the tangent and normal spaces to $M$ at $x$, $A_N = A^G/N_x$, $C = (\delta h_j/\delta x)(x, y)$ and $D = (\delta h_j/\delta y)(x, y)$. Since $G(x + y) \subset \Omega_H$ is an isolated orbit in the zero set of a normal map $f_3$, we can compute $\deg_H(f_3, \Omega)$ by definition. Let $w$ be the unit tangent vector to $G(x + y)$ at $(x + y)$ (as in 4.12(5)). We observe that

$$\text{sgn}(Df_3(x + y), w) = \text{sgn}(A^G(x, v) \cdot \text{sgn}(p_j), (−1)).$$

Since $\text{sgn}(A^G(x), v) = −1$, we have $\deg_H(f_3, \Omega) = p_j$. The proof of Theorem 4.9 is complete.
5. $S^1$-degree in Banach spaces. In this section we will extend the results of the preceding sections to the case of $G$-equivariant maps between subsets of Banach $G$-spaces ($G = S^1$). Our approach imitates closely the Leray–Schauder extension of the classical degree theory.

5.1. Definition. Let $E$ be a Banach space and let $GL(E)$ denote the group of all its linear automorphisms. $E$ is called a Banach $G$-space or a representation of $G$ if there is given a homomorphism $q: G \to GL(E)$ such that the map $\mu: G \times E \to E$ defined by $\mu(g, x) = q(g)x$ is continuous. Note that, by the Banach–Steinhaus theorem, assuming the continuity of $q$ as a map from $G$ into $GL(E)$ equipped with the strong topology, one obtains an equivalent definition of a Banach $G$-space.

Note also that since $G = S^1$ is compact, we may assume that $q$ satisfies the following condition:

\[ (*) \quad q(g): E \to E \text{ is an isometry for every } g \in G. \]

Indeed, if $\| \cdot \|$ is a norm in $E$ then the formula

\[ \|x\|_* = \int \|q(g)x\| d\mu, \]

where $\mu$ denotes the normalized Haar measure on $G$, defines an equivalent norm such that $\|q(g)x\|_* = \|x\|_*$ for all $g \in G$.

In what follows we assume that $E$ is a Banach $G$-space with a norm $\| \cdot \|$ satisfying $(*)$. We denote by $E \oplus R$ the direct product of $E$ and $R$, i.e. $E \oplus R$ is a Banach $G$-space by $g(x, t) = (gx, t)$. We also let $Q: E \oplus R \to E$ denote the projection.

Recall that if $X$ is a topological space then a continuous map $f: X \to E$ is called compact if the closure of $f(X)$ is a compact subset of $E$.

5.2. Definition. Let $E$ be a Banach $G$-space and let $X$ be a topological $G$-space. A continuous map $f: X \to E$ is called a compact $G$-map if it is $G$-equivariant and compact. We say that a continuous map $h: X \times [0, 1] \to E$ is a compact $G$-homotopy if it is compact and $h(gx, t) = gh(x, t)$ for all $x \in X$, $g \in G$ and $t \in [0, 1]$.

Our aim is to define a degree for $S^1$-equivariant maps of the form $Q + f: \Omega \to E$, where $\Omega$ is an open bounded invariant subset of $E \oplus R$ and $f$ is a compact $G$-map such that $(Q + f)(\partial \Omega) = E \backslash \{0\}$.

5.3. Theorem. Let $Q$ run through the family of all bounded invariant subsets of $E \oplus R$ and $f: X \to E$ through compact $G$-maps such that $X \subset E \oplus R$ is invariant, $\Omega \subset X$ and $(Q + f)(\partial \Omega) \subset E \backslash \{0\}$. Then there exists an $\mathbb{A}$-valued function $\text{Deg}(Q + f, \Omega)$, called the $S^1$-degree, satisfying the following conditions:

(a) If $\text{deg}_H(Q + f, \Omega) \neq 0$ then $(Q + f)^{-1}(0) \cap \Omega^H \neq \emptyset$.

(b) If $\Omega_0 \subset \Omega$ is open, invariant and $(Q + f)^{-1}(0) \cap \Omega \subset \Omega_0$ then $\text{Deg}(Q + f, \Omega) = \text{Deg}(Q + f, \Omega_0)$. 
(c) If $\Omega_1, \Omega_2$ are open invariant subsets of $\Omega$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $(Q + f)^{-1}(0) \cap \Omega = \Omega_1 \cup \Omega_2$ then $\text{Deg}(Q + f, \Omega) = \text{Deg}(Q + f, \Omega_1) + \text{Deg}(Q + f, \Omega_2)$.

(d) If $h: \bar{\Omega} \times [0, 1] \to E$ is a compact G-homotopy such that $x + h(x, t) \neq 0$ for all $(x, t) \in \partial \Omega \times [0, 1]$ then $\text{Deg}(Q + h_0, \Omega) = \text{Deg}(Q + h_1, \Omega)$.

(e) Suppose $E = E_1 \oplus E_2$ and $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1$ (resp. $\Omega_2$) is an open bounded invariant subset of $E_1$ (resp. $E_2 \oplus R$). Let $Q_2: E_2 \oplus R \to E_2$ denote the projection. Suppose further that $0 \in \Omega_1$ and $f_2: \bar{\Omega}_2 \to E_2$ is a compact map such that $Q_2(y) + f_2(y) \neq 0$ for all $y \in \partial \Omega_2$. Define $\bar{f}: \bar{\Omega} \to E$ by $f(x, y) = f_2(y)$, where $(x, y) \in \bar{\Omega} = \bar{\Omega}_1 \times \bar{\Omega}_2 \subset E_1 \oplus (E_2 \oplus R)$. Then $\text{Deg}(Q + f, \Omega) = \text{Deg}(Q_2 + f_2, \Omega_2)$.

Recall that a $G$-map $f: X \to E$, where $X$ is a $G$-space, is called finite-dimensional if there exists a finite-dimensional subspace $L \subset E$ such that $f(X) \subset L$.

The proof of Theorem 5.3 is based on the following approximation lemma.

5.4. Lemma. Let $X$ be a bounded closed invariant subset of a Banach $G$-space $F$. A $G$-map $f: X \to E$ is compact if and only if it is the limit of a uniformly convergent sequence of finite-dimensional $G$-maps.

Proof. The condition is evidently sufficient. Our proof of necessity follows the classical lines (cf. Dugundji–Granas [9]). We will prove that for every $\epsilon > 0$ there exists a $G$-equivariant finite-dimensional $\epsilon$-approximation of $f$. For a given $\epsilon > 0$ choose an $\epsilon$-net $\{y_1, \ldots, y_k\}$ for $f(X)$. Since the set of elements of $E$ whose orbits lie in a finite-dimensional subspace of $E$ is dense in $E$ (Mostow [18]), we may assume that $g y_i \subset V_i$, $i = 1, \ldots, k$, where $V_i$ is a finite-dimensional linear $G$-subspace of $E$. Thus there exists a continuous map $\varphi_\epsilon: X \to V = V_1 + \ldots + V_k$ such that $\|f(x) - \varphi_\epsilon(x)\| < \epsilon$ for all $x \in X$. Set

$$f_\epsilon(x) = \int_0^1 g \varphi_\epsilon(\epsilon^{-1} x) \, d\mu.$$  

Clearly $f_\epsilon$ is a $G$-map, $f_\epsilon(X) \subset V$ and $\|f(x) - f_\epsilon(x)\| < \epsilon$ for all $x \in X$. The proof is complete.

Suppose now that $\Omega$ is an open bounded invariant subset of $E \oplus R$, and $\bar{f}: \bar{\Omega} \to E$ is a compact $G$-map such that $(Q + f)(\partial \Omega) \subset E \setminus \{0\}$. Since $(Q + f)(\partial \Omega)$ is closed in $E$, we have $\eta = \inf \{|Q(x) + f(x)|: x \in \partial \Omega\} > 0$. By 5.4 we can find a $G$-invariant finite-dimensional linear subspace $V \subset E$ and a $G$-map $f_\epsilon: \bar{\Omega} \to E$ such that $f_\epsilon(\bar{\Omega}) \subset V$ and $\|f(x) - f_\epsilon(x)\| < \eta$ for all $x \in \bar{\Omega}$. Thus $Q + f_\epsilon: (\bar{\Omega}, \partial \Omega) \to (E, E \setminus \{0\})$ and $(Q + f_\epsilon)(\bar{\Omega} \cap V \oplus R) \subset V$. Therefore, by Section 3, $\text{Deg}(Q + f_\epsilon, \bar{\Omega} \cap V \oplus R)$ is defined.

5.5. Definition. $\text{Deg}(Q + f, \Omega) = \text{Deg}(Q + f_\epsilon, \Omega \cap V \oplus R)$, where $V$ and $f_\epsilon$ are as above.

Using Lemma 5.4, the homotopy invariance and the contraction property of the $S^1$-degree in the finite-dimensional case (1.2(d) and (e)), one can check
that the definition of $\text{Deg}(Q + f, \Omega)$ does not depend on the choice of $f$ and $V$. The argument is the same as in the case of the Leray–Schauder degree.

**Proof of Theorem 5.3.** To prove (a) take a sequence $\varepsilon_n$ such that \( \lim_{n} \varepsilon_n = 0 \) and $\varepsilon_n < \inf \{ ||Q(x) + f(x)|| : x \in \partial \Omega \}$. By 5.4 there exists a sequence \( \{f_n\}_n \), where $f_n : \bar{\Omega} \to V$ is a $G$-map, $V_n$ is a finite-dimensional linear $G$-subspace of $E$ and $f_n$ is an $\varepsilon_n$-approximation of $f$. By 5.5, $\text{Deg}(Q + f_n, \Omega \cap V_n \oplus R) = \text{Deg}(Q + f, \Omega) \neq 0$. Therefore, by 1.2(a), for every $n$ there exists $x_n = (y_n, t_n) \in \Omega^H$ such that $y_n + f_n(y_n, t_n) = 0$. Since $f(\bar{\Omega})$ is compact, we may assume that \( \{f(x_n)\} \) is convergent. Then

\[
||y_n - y_m|| = ||f_n(x_n) - f_m(x_m)|| \leq ||f(x_n) - f(x_m)|| + \varepsilon_n + \varepsilon_m
\]

and thus \( \{y_n\} \) converges. We may assume that \( (y_n, t_n) \to (y_0, t_0) \). Therefore $y_0 + f(y_0, t_0) = 0$. This completes the proof of (a).

(b) and (c) are easy consequences of 1.2(b) and (c), respectively. To prove (d) suppose that $h : \bar{\Omega} \times [0, 1] \to E$ is a compact homotopy such that $Q(x) + h(x, t) = 0$ for all $(x, t) \in \partial \Omega \times [0, 1]$. Given $\varepsilon > 0$, using again 5.4, we find a finite-dimensional $\varepsilon$-approximation $h : \bar{\Omega} \times [0, 1] \to V$ such that $Q(x) + h(x, t) \neq 0$ for all $(x, t) \in \partial \Omega \times [0, 1]$. Then, by 5.5 and 1.2(d),

\[
\text{Deg}(Q + h_0, \Omega) = \text{Deg}(Q + (h_0)_1, \Omega \cap V \oplus R) = \text{Deg}(Q + (h_1)_1, \Omega \cap V \oplus R) = \text{Deg}(Q + h_1, \Omega).
\]

Finally, (e) follows easily from Definition 5.5 and 1.2(e), which completes the proof of Theorem 5.3.

With a view to applications discussed in the last sections it is convenient to extend the definition of the $S^1$-degree to a slightly larger class of maps. Consider two Banach $G$-spaces $E$ and $F$ together with a linear $G$-isomorphism $T : E \to F$. Let $\Omega \subset E \oplus R$ be an open $G$-invariant bounded set. We will consider $G$-maps of the form $TQ + f : \bar{\Omega} \to F$ such that $f$ is compact, where $Q : E \oplus R \to E$ denotes the projection.

**5.6. DEFINITION.** Suppose that $\Omega$ is an open $G$-invariant bounded subset of $E \oplus R$ and $f : \bar{\Omega} \to F$ is a compact $G$-map such that $(TQ + f)(\partial \Omega) \subset F \setminus \{0\}$. Define the $S^1$-degree of $TQ + f$ by $\text{Deg}(TQ + f, \Omega, T) = \text{Deg}(Q + T^{-1}f, \Omega)$.

As a direct consequence of Definition 5.6 we obtain the following reformulation of Theorem 5.3.

**5.7. THEOREM.** Let $\Omega$ run through the family of all open bounded invariant subsets of $E \oplus R$ and $f : X \to F$ through compact $G$-maps such that $X \subset E \oplus R$ is invariant, $\bar{\Omega} \subset X$ and $(TQ + f)(\partial \Omega) \subset F \setminus \{0\}$. Then there exists an $\omega$-valued function $\text{Deg}(TQ + f, \Omega, T)$, called the $S^1$-degree, satisfying the following conditions:

(a) If $\deg_{\Omega}(TQ + f, \Omega, T) \neq 0$ then $(TQ + f)^{-1}(0) \cap \Omega^H \neq \emptyset$.

(b) If $\Omega_0 \subset \Omega$ is open, invariant and $(TQ + f)^{-1}(0) \cap \Omega \subset \Omega_0$ then $\text{Deg}(TQ + f, \Omega, T) = \text{Deg}(TQ + f, \Omega_0, T)$. 

(c) If $\Omega_1, \Omega_2$ are open invariant subsets of $\Omega$ such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $(TQ + f)^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$ then \( \text{Deg}(TQ + f, \Omega, T) = \text{Deg}(TQ + f, \Omega_1, T) + \text{Deg}(TQ + f, \Omega_2, T) \).

(d) If $h : \bar{\Omega} \times [0, 1] \to F$ is a compact $G$-homotopy such that $TQ(x) + h(x, t) \neq 0$ for all $(x, t) \in \partial \Omega \times [0, 1]$ then \( \text{Deg}(TQ + h_0, \Omega, T) = \text{Deg}(TQ + h_1, \Omega, T) \).

(e) Suppose $E = E_1 \oplus E_2$. For $i = 1, 2$ set $F_i = T(E_i)$ and let $T_i : E_i \to F_i$ denote the restriction. Suppose also that $\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1$ (resp. $\Omega_2$) is an open bounded invariant subset of $E_1$ (resp. $E_2 \oplus R$). Assume further that $0 \in \Omega_1$ and $f_2 : \bar{\Omega}_2 \to F_2$ is a compact map such that $TQ(y) + f_2(y) \neq 0$ for all $y \in \partial \Omega_2$.

Define $f : \bar{\Omega} \to \bar{F}$ by $f(x, y) = f_2(y)$, where $(x, y) \in \Omega = \Omega_1 \times \Omega_2 \subset E_1 \oplus (E_2 \oplus R)$. Then \( \text{Deg}(TQ + f, \Omega, T) = \text{Deg}(TQ + f_2, \Omega_2, T_2) \), where $Q_2 : E_2 \oplus R \to E_2$ denotes the projection.

6. Computations in Banach spaces. Let $E$ denote a real infinite-dimensional Banach space. We will use the following notations:

\[
L(E) = \text{the Banach algebra of all linear and continuous operators from } E \text{ into itself,}
I \in L(E) = \text{the identity operator,}
K(E) = \text{the ideal of compact operators in } L(E),
GL(E) = \text{the group of all invertible operators in } L(E),
L_K(E) = \{ A \in L(E) : A = I + B, B \in K(E) \},
GL_C(E) = GL(E) \cap L_K(E).
\]

If $E_1, E_2$ are closed subspaces of $E$ such that $E = E_1 \oplus E_2$ (i.e. $E_1$ and $E_2$ are complementary) and $A \in L(E)$ then the decomposition $E = E_1 \oplus E_2$ determines a matrix

\[
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix},
\]

where $A_{ij} : E_j \to E_i$, $i, j = 1, 2$.

6.1. DEFINITION. Let $B \in K(E)$. A real number $\mu$ is called a characteristic value of $B$ if $\text{Ker}(I - \mu B) \neq \{0\}$.

Note that $\mu$ is a characteristic value of $B$ if and only if $\mu \neq 0$ and $\mu^{-1}$ is an eigenvalue of $B$. Therefore by the Riesz–Schauder theorem, any closed interval contains only a finite number of characteristic values of $B$. For a characteristic value $\mu$ we let $L(\mu) = \bigcup \text{Ker}(I - \mu B)^k$; $L(\mu)$ is called the generalized kernel of $I - \mu B$. It is known that $\dim L(\mu) < \infty$.

Suppose now that $A = I - B \in GL_C(E)$. Denote by $\{ \mu_1 < \ldots < \mu_d \}$ the set of all characteristic values of $B$ contained in $[0, 1]$. Let $L = L(\mu_1) + \ldots + L(\mu_d)$ ($L = \{0\}$ if $B$ has no characteristic value in $[0, 1]$) and set $\text{sgn } A = (-1)^d$, where $d = \dim L$. The following proposition collects standard properties of $\text{sgn } A$. 
6.2. Proposition (cf. Kato [16]).

(a) \(\text{GL}_C(E)\) has two connected components. \(A_1, A_2 \in \text{GL}_C(E)\) are in the same connected component if and only if \(\text{sgn}A_1 = \text{sgn}A_2\).

(b) If \(E = E_1 \oplus E_2\) and \(A \in \text{GL}_C(E)\) is of the form

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},
\]

then \(\text{sgn}A = \text{sgn}A_{11} \text{sgn}A_{22}\).

Consider now the Banach space \(E \oplus \mathbb{R}\) and let \(Q: E \oplus \mathbb{R} \to E\) denote the projection. Let \(L(E \oplus \mathbb{R}, E)\) denote the Banach space of linear operators from \(E \oplus \mathbb{R}\) into \(E\). Let

\[
K(E \oplus \mathbb{R}, E) = \text{the subspace of compact operators in } L(E \oplus \mathbb{R}, E),
\]

\[
\text{SL}_C(E) = \{ A \in L(E \oplus \mathbb{R}, E): A(E \oplus \mathbb{R}) = E \text{ and } A = Q + B, B \in K(E \oplus \mathbb{R}, E) \}.
\]

Suppose now that \(A \in \text{SL}_C(E)\). Since \(A\) is a Fredholm operator of index 1 and \(A\) is surjective, \(\dim \ker A = 1\). Choose \(v \in \ker A\) such that \(v \neq 0\) (note that this determines an orientation of \(\ker A\)). Let \(\xi: E \oplus \mathbb{R} \to \mathbb{R}\) be a linear functional such that \(\xi(v) = 1\). Define \(A^- : E \oplus \mathbb{R} \to E \oplus \mathbb{R}\) by \(A^-(x) = (A(x), \xi(x))\). Clearly \(A^- \in \text{GL}_C(E \oplus \mathbb{R})\). Set

\[
\text{sgn}(A, v) = \text{sgn}A^-
\]

It is easy to check that the above definition is correct and independent of the choice of \(\xi\).

From 6.2 we obtain at once

6.3. Proposition. If \(A \in \text{SL}_C(E)\), \(E = E_1 \oplus E_2\), \(v \in \ker A \subset E_2\), \(v \neq 0\) and, with respect to the decompositions \(E \oplus \mathbb{R} = E_1 \oplus E_2 \oplus \mathbb{R}\), \(E = E_1 \oplus E_2\), \(A\) has the representation

\[
A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad E_1 \oplus \mathbb{R} \to \oplus \quad E_2 \oplus \mathbb{R} \to E_2
\]

then \(\text{sgn}(A, v) = \text{sgn}A_{11} \text{sgn}(A_{22}, v)\).

Consider now two Banach spaces \(E, F\) together with an isomorphism \(T: F \to E\). Let \(Q: F \oplus \mathbb{R} \to E\) be defined by \(Q(y, t) = T(y)\). Let \(K(F, E)\) (resp. \(K(F \oplus \mathbb{R}, E)\)) denote the space of all compact linear maps from \(F\) (resp. \(F \oplus \mathbb{R}\)) into \(E\). Set

\[
\text{GL}_C(T) = \{ A: A = T + A_0: F \to E \text{ is an isomorphism, } A_0 \in K(F, E) \},
\]

\[
\text{SL}_C(T) = \{ A: A = Q + A_0: F \oplus \mathbb{R} \to E, A_0 \in K(F \oplus \mathbb{R}, E), A(F \oplus \mathbb{R}) = E \}.
\]
For $A \in \text{GL}_c(T)$ we set
\[ \text{sgn}_T A = \text{sgn}(T^{-1} A). \]

Take $B \in \text{SL}_c(T)$; obviously $T^{-1} B \in \text{SL}_c(F)$. As in the case $T = I$ we have $\dim \text{Ker} B = 1$. For $\nu \in \text{Ker} B$, $\nu \neq 0$, we set
\[ \text{sgn}_T (B, \nu) = \text{sgn}(T^{-1} B, \nu). \]

As a direct consequence of the above definitions, we have

**6.4. Proposition.** Suppose $A \in \text{GL}_c(T)$, $B \in \text{SL}_c(T)$, $\nu \in \text{Ker} B$, $\nu \neq 0$.

(a) $T \in \text{GL}_c(A)$ and $\text{sgn}_T A = \text{sgn}_A T$.

(b) If $\eta: [0, 1] \to \text{GL}_c(T)$ is continuous, then $\text{sgn}_T \eta(0) = \text{sgn}_T \eta(1)$.

(c) $\text{sgn}_T (B, \nu) = \text{sgn}_T A \text{sgn}_A (B, \nu)$.

Suppose now $F = F_1 \oplus F_2$ and let $E_i = T(F_i)$, $i = 1, 2$. Clearly $E = E_1 \oplus E_2$. Let $T_i: F_i \to E_i$, $i = 1, 2$, denote the corresponding restrictions. Suppose further that $B \in \text{SL}_c(T)$ and, with respect to the decompositions $F \oplus R = F_1 \oplus F_2 \oplus R$, $E = E_1 \oplus E_2$,
\[ B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \]
and $\nu \in \text{Ker} B \subset F_2 \oplus R$. Then we have

**6.5. Proposition.** $\text{sgn}_T (B, \nu) = \text{sgn}_{T_1}(B_{11}) \cdot \text{sgn}_{T_2}(B_{22}, \nu)$.

Suppose now that we have isomorphisms of Banach spaces:

\[ T: F \to E, \quad R: F_1 \to E_1, \quad \Phi: F \oplus R \to F_1 \oplus R, \quad \Psi: E \to E_1. \]

Suppose further that $RQ_1, \Phi = \Psi TQ$, where $Q$, $Q_1$ denote the corresponding projections.

**6.6. Proposition.** Assume that $A \in \text{SL}_c(T)$ and $x \in \text{Ker} A$, $x \neq 0$. Then $\Psi A \Phi^{-1} \in \text{SL}_c(R)$ and $\text{sgn}_R(\Psi A \Phi^{-1}, \Phi(x)) = \text{sgn}_T(A, x)$.

Now we are ready to formulate an infinite-dimensional generalization of Theorem 4.1. As above, we consider two Banach $G$-spaces $E$, $F$ together with a $G$-equivariant isomorphism $T: F \to E$. Suppose that $\Omega \subset F \oplus R$ is bounded, open, invariant and isometric: $\bar{\Omega} \to E$ is a $C^1$-map of the form $f(x, p) = Tx + \Phi(x, p)$, where $\Phi$ is a compact $G$-map. Suppose further that there exists $a \in \Omega$ such that $G_a = Z_k \not= S^1$, $f^{-1}(0) = Ga \subset \Omega$ and $Df(a): F \oplus R \to E$ is surjective. Let $K = Z_k$ and let $f^K: \bar{\Omega}^K \to E^K$ denote the restriction of $f$. Clearly $Df^K(a): F^K \oplus R \to E^K$ is also surjective. Let $v$ denote the tangent vector to the orbit $Ga$ at $a$. Notice that $v \in \text{Ker} Df^K(a)$. Finally, let $R = T^K: F^K \to E^K$ denote the restriction of $T$. 
6.7. Theorem. With the above assumptions we have:

(a) If \( \text{sgn}_T(Df(a), v) = \text{sgn}_K(Df^K(a), v) \) then

\[
\text{deg}_H(f, \Omega, T) = \begin{cases} 
\text{sgn}_K(Df^K(a), v) & \text{if } H = K = Z_K, \\
0 & \text{if } H \neq K.
\end{cases}
\]

(b) If \( \text{sgn}_T(Df(a), v) = -\text{sgn}_K(Df^K(a), v) \) then \( k \) is even and, letting \( k = 2m \), we have

\[
\text{deg}_H(f, \Omega, T) = \begin{cases} 
\text{sgn}_K(Df^K(a), v) & \text{if } H = K = Z_K, \\
-\text{sgn}_K(Df^K(a), v) & \text{if } H = Z_m, \\
0 & \text{if } H \neq Z_K \text{ and } H \neq Z_m.
\end{cases}
\]

We shall use a linear slice in the proof of 6.7.

6.8. Theorem. Let \( E \) be a Banach representation of \( G = S^1 \), \( x_0 \) a point of \( E \), and \( K \) the isotropy group of \( x_0 \). Then there exists a \( K \)-equivariant linear subspace \( N \subset E \) (of codimension equal to the dimension of \( G/K \)), a ball \( B_s(N) \) in \( N \) with center at \( x_0 \) and an open neighborhood \( U \) of \( Gx_0 \) such that the map \( \eta: G \times K B_s(N) \to U \) given by \( \eta([g, v]) = g v \) is a \( G \)-homeomorphism.

(See Jodel–Marzantowicz [15] for the proof of this theorem for any compact Lie group.)

Proof of Theorem 6.7. First we observe that \( \Omega \) can be replaced by the open tube \( U \approx G \times K B_s(N) \) given by 6.8. For a given \( G \)-map \( \Psi \) from \( U \) into some \( Y \) we denote by \( \Psi_N \) its restriction to \( U_N = \{e\} \times K B_s(N) \subset N \). Obviously \( \Psi_N \) is \( K \)-equivariant. As in 4.2 the assignment \( \Psi \mapsto \Psi_N \) gives a one-to-one correspondence between \( G \)-maps from \( (U, \partial U) \) into \( (Y, Y \setminus \{0\}) \) and \( K \)-maps from \( (U_N, \partial U_N) \) into \( (Y, Y \setminus \{0\}) \) for any \( G \)-representation \( Y \). The inverse assignment is \( \Psi_N \mapsto G \times K \Psi_N \), where \( G \times K \Psi_N([g, v]) = g \Psi_N(v) \). The same assignment gives a one-to-one correspondence between equivariant homotopy classes too.

In order to derive \( \text{deg}_E : (f, U, T) \to \text{deg}_S : (T^{-1} \circ f, U) \).

Note that \( T^{-1} \circ f = Q + T^{-1} \circ \Phi \), where \( Q : F \oplus R \to F \) is the projection. Let \( \Psi = T^{-1} \circ f \); then \( D\Psi(x_0) = Q + T^{-1} \circ D\Phi(x_0) \) is \( K \)-equivariant, so the formula \( \Psi^\wedge(x) = D\Psi(x_0)(x - x_0) \) defines a \( K \)-equivariant map. Note also that since \( Gx_0 = M = f^{-1}(0) \cap U \), \( M \) is a smooth manifold of dimension 1 and there exists a unique tangent vector \( x'_0 \). Moreover, \( N \) is a normal space at \( x'_0 \), i.e. \( N \oplus \text{span}(x'_0) = F \oplus R \). From a Rouche principle argument applied to the isomorphism \( \Psi \) it follows that there exists an open \( K \)-invariant subset \( U_\varepsilon \subset U_N \) such that \( \Psi_N \) and \( \tilde{\Psi}_N \) are \( K \)-homotopic as maps of pairs \( (U_\varepsilon, \partial U_\varepsilon) \to (F, F \setminus \{0\}) \). As a consequence, \( \Psi \) is \( G \)-homotopic to \( G \times K \tilde{\Psi}_N \) on \( GU_\varepsilon \).

Next, we \( G \)-homotopically replace \( G \times K \tilde{\Psi}_N \) by a \( G \)-map which is a finite-dimensional perturbation of \( Q \).
6.9. **Lemma.** There exists a $K$-homotopy $\eta: [0, 1] \times N \to F$ such that:

(a) for every $t$ the map $\eta(t, \cdot)$ is $K$-equivariant, compact and linear,
(b) for every $t$ the map $Q + \eta(t, \cdot)$ is an isomorphism,
(c) $\eta(0, \cdot) = T^{-1}Df_N(x_0)$,
(d) $\eta(1, \cdot)(N) \subset V$, where $V$ is a $G$-invariant finite-dimensional subspace of $F$.

**Proof.** The lemma can be proved by a modification of nonequivariant arguments of the proof of the fact that a compact linear perturbation of identity can be deformed through isomorphisms into a finite-dimensional perturbation of identity.

After this preparation we are able to complete the proof of Theorem 6.7. Denote $GU_x$ by $\Omega_1$. Observe first that we can assume $M = \Psi^{-1}(0) \cap \Omega_1$ to be a subset of a finite-dimensional subrepresentation $V_1 \subset F \oplus R$, where $V_1 = W \oplus R$. (Since $G$ is abelian, $(F \oplus R)^K$ is a subrepresentation of $F \oplus R$.) Let $y \in (F \oplus R)^K$ be a vector having a finite-dimensional orbit which is close enough to $x_0$. Since $U_N$ is a slice there exists $g \in G$ such that $y_0 = gy \in U_N$. It is easy to check that the map $\Psi(x) = G \times _K \Psi_N(x + x_0 - y_0)$ is $G$-homotopic to $\Psi$, and $\Psi^{-1}(0) \cap \Omega_1 = G y_0$. Let $V_1 \subset F$ be the subrepresentation given by 6.9, and $V = V_1 + V_2$. Let $\Psi^* = \Omega_N + \eta(1, \cdot)$ where $\eta$ is given by 6.9. From 6.9 and 6.4(b) it follows that

$$
\text{sgn}(D\Psi(x_0), x_0) = \text{sgn}(D(G \times _K \Psi^*)(x_0), x_0),
$$

$$
\text{sgn}(D\Psi^K(x_0), x_0) = \text{sgn}(D(G \times _K \Psi^*)(x_0), x_0),
$$

since $x_0 \in \text{Ker} D\Psi(x_0) = \text{Ker} D(G \times _K \Psi^*)(x_0) \subset (F \oplus R)^K$. By the properties of the $S^1$-degree,

$$
\text{Deg}(\Psi, \Omega_1) = \text{Deg}(G \times _K \Psi^*, \Omega_1).
$$

Since $G \times _K \Psi^* = Q + \Phi^*$ with $\Phi^*(\Omega_1) \subset V$, from Definition 5.5 it follows that

$$
\text{Deg}(\phi|_\nu, \Omega_\nu) = \text{Deg}(G \times _K \Psi^*, \Omega_1),
$$

where $\Omega_\nu = \Omega_1 \cap V$ and $\phi$ is the restriction of $G \times _K \Psi^*$ to $\Omega_\nu$. Let next $F_1$ be a subspace of $F$ such that $F_1 \oplus V = F$. With respect to this decomposition $D(G \times _K \Psi^*)(x_0)$ has the form

$$
D(G \times _K \Psi^*)(x_0) = \begin{bmatrix} D\phi(x_0) & S \\ 0 & \text{id} \end{bmatrix}.
$$

From 6.3 it follows that

$$
\text{sgn}(\phi'(x_0), x_0) = \text{sgn}(D(G \times _K \Psi^*)(x_0), x_0),
$$

$$
\text{sgn}(\phi'(x_0)^K, x_0) = \text{sgn}(D(G \times _K \Psi^*)(x_0), x_0).
$$

Now Theorem 4.1 applied to $(\phi|_\nu, \Omega_\nu)$ gives the conclusion of Theorem 6.7.
7. $S^1$-degree and periodic solutions of ODE. Throughout this section we shall assume that $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ is a $C^\infty$ map. Consider the autonomous differential equation

$$(*) \quad y'(t) = \varphi(y(t)).$$

For each $v \in \mathbb{R}^n$ there exists a unique solution of $(*)$, $y(t) = \eta(v, t)$, satisfying $y(0) = \eta(v, 0) = v$ and defined in an open $t$-interval $I_v$ containing $0$. We will denote by $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ the union of the sets $\{v\} \times I_v$, where $v \in \mathbb{R}^n$. Clearly $\Omega$ is open and $\eta: \Omega \to \mathbb{R}^n$ is a $C^\infty$ mapping which satisfies, wherever defined, the group property

$$\eta(\eta(v, s), t) = \eta(v, s + t).$$

Set

$$\Omega_+ = \{(v, t) \in \Omega: t > 0\}, \quad \Sigma = \{(v, p) \in \Omega_+: \eta(v, p) = v\}.$$

Recall that $v \in \mathbb{R}^n$ is called a stationary point of $(*)$ if $\varphi(v) = 0$. We let

$$\text{Sing}(\varphi) = \{v \in \mathbb{R}^n: \varphi(v) = 0\};$$

note that $\text{Sing}(\varphi) \times (0, \infty) \subset \Sigma$.

Define the nonlinear Fuller map associated with the equation $(*)$, $\pi: \Omega \to \mathbb{R}^n$, by $\pi(w, \lambda) = w - \eta(w, \lambda)$. We let $P: \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R}^n$ denote the linear map $D\pi(v, p)$ and call it the linear Fuller map associated with $(*)$ at $(v, p)$.

We say that $v \in \mathbb{R}^n$ is a $p$-periodic point with respect to $(*)$ if $(v, p) \in \Sigma$. If $v$ is a nonstationary $p$-periodic point then there is the least $q > 0$ such that $\eta(v, q) = v$ and $p = mq$. We call $p$ the period, $q$ the least period, and $m = q^{-1}p$ the multiplicity of $(v, p)$.

Suppose now that $v \in \mathbb{R}^n$ is nonstationary and $p$-periodic. Let $P$ denote the linear Fuller map of $(*)$ at $(v, p)$ and let $A: \mathbb{R}^n \to \mathbb{R}^n$ denote the linear map $D\eta_p(v)$, where $\eta_p(w) = \eta(w, p)$. Since $\eta_p$ is a local diffeomorphism, $A$ is a linear automorphism of $\mathbb{R}^n$. Moreover, $A(\varphi(v)) = \varphi(v)$ and thus $1$ belongs to the spectrum $\sigma(A)$ of $A$. We say that $v$ is elementary if the linear subspace of $\mathbb{R}^n$ corresponding to the eigenvalue $1$ is one-dimensional and $A$ has no eigenvalue of absolute value $1$ different from $1$ (note that $v$ is elementary if the periodic orbit which starts from $v$ is hyperbolic). Let $L$ denote the one-dimensional linear subspace spanned by $\varphi(v)$ and $N$ the linear subspace corresponding to $\sigma(A) \setminus \{1\}$. Clearly $\mathbb{R}^n = N \oplus L$. Note also that $P(N) = N$ and $P(0, 1) = -\varphi(v)$.

The above shows that if $v$ is elementary then $P(\mathbb{R}^n \oplus \mathbb{R}) = \mathbb{R}^n$.

Together with $(*)$ we will consider the equation

$$(**) \quad x'(t) = p\varphi(x(t)).$$

As a direct consequence of the introduced definitions we have:

7.1. Remark. Let $(v, p) \in \mathbb{R}^n \times (0, \infty)$. Then $(v, p) \in \Omega_+$ if and only if equation $(**)$ has a solution $x: [0, 1] \to \mathbb{R}^n$ satisfying the initial condition $x(0) = v$, and the solution is given by $x(t) = \eta(v, pt)$.
We will use the following notations:

\[ C^1 = C^1([0, 1], \mathbb{R}^n) = \text{the Banach space of } C^1\text{-functions } x: [0, 1] \rightarrow \mathbb{R}^n \text{ with the standard norm}, \]

\[ C = C([0, 1], \mathbb{R}^n) = \text{the Banach space of continuous functions } x: [0, 1] \rightarrow \mathbb{R}^n \text{ with the standard norm}, \]

\[ E^1 = \{ x \in C^1: x(0) = x(1), x'(0) = x'(1) \}, \]

\[ E = \{ x \in C: x(0) = x(1) \}, \]

\[ S: C^1 \rightarrow C \oplus \mathbb{R}^n, \quad S(x) = (x' + x, x(0) - x(1)), \]

\[ T: E^1 \rightarrow E, \text{ the restriction of } S. \]

Clearly, both \( S \) and \( T \) are isomorphisms of Banach spaces.

For \( x: [0, 1] \rightarrow \mathbb{R}^n \) and \( \theta = e^{2n i \theta} \in G = S^1 = \{ z \in C: |z| = 1 \} \) define

\[ (gx)(t) = \begin{cases} x(t + \theta) & \text{for } t + \theta \leq 1, \\ x(t + \theta - 1) & \text{for } t + \theta > 1. \end{cases} \]

With the above-defined \( S^1 \) action \( E^1 \) and \( E \) are \( S^1 \)-Banach spaces and \( T \) is an equivariant isomorphism. Define \( f: E^1 \oplus \mathbb{R} \rightarrow E \) by \( f(x, \lambda) = x' - \lambda \varphi(x(\cdot)) \). The following is evident:

7.2. Remark. \( f \) is equivariant and \( f - T \) is completely continuous.

The following theorem states the main result of this section:

7.3. Theorem. Suppose that \( v \in \mathbb{R}^n \) is an elementary \( p \)-periodic point and \( (v, p) \) is of multiplicity \( m \) with respect to equation (\( * \)) and define \( \xi \in E^1 \) by \( \xi(t) = \eta(v, pt) \). Then there exists an open bounded subset \( \mathcal{U} \subset E^1 \times (0, \infty) \) such that \( f^{-1}(0) \cap \mathcal{U} = G\xi, p \) and \( f(x, \lambda) \neq 0 \) for all \( (x, \lambda) \in \partial \mathcal{U} \) and we have

(a) if \( \text{sgn}(P, \varphi(v)) = \text{sgn}(P_m, \varphi(v)) \) then

\[ \deg_H(f, \mathcal{U}, T) = \begin{cases} \text{sgn}(P, \varphi(v)) & \text{if } H = \mathbb{Z}_m, \\ 0 & \text{if } H \neq \mathbb{Z}_m. \end{cases} \]

(b) if \( \text{sgn}(P, \varphi(v)) = -\text{sgn}(P_m, \varphi(v)) \) then \( m \) is even and letting \( m = 2\mu \) we have

\[ \deg_H(f, \mathcal{U}, T) = \begin{cases} -\text{sgn}(P, \varphi(v)) & \text{if } H = \mathbb{Z}_m, \\ \text{sgn}(P, \varphi(v)) & \text{if } H = \mathbb{Z}_m, \\ 0 & \text{if } H \neq \mathbb{Z}_m, H \neq \mathbb{Z}_m, \end{cases} \]

where \( P_m \) denotes the linear Fuller map of (\( * \)) at \( (v, q) \) and \( P \) denotes the linear Fuller map of (\( * \)) at \( (v, p) \).

We postpone the proof of Theorem 7.3 to the end of this section. First we will prove the following result which plays a crucial role in our argument.
7.4. **Proposition.** Suppose that \( v \in \mathbb{R}^n \) is an elementary \( p \)-periodic point with respect to equation (\( * \)). Define \( \xi \in E^1 \) by \( \xi(t) = \eta(v, pt) \). Then \( Df(\xi, p) \) is surjective, \( \xi \in \text{Ker} Df(\xi, p) \) and

\[
\text{sgn}_P(Df(\xi, p), \xi') = \text{sgn}(P, \varphi(v)),
\]

where \( P \) denotes the linear Fuller map of (\( * \)) at \( (v, p) \).

Before proving 7.4 we have to prove auxiliary results. We use the following notation:

\[
f^\wedge: C^1 \oplus \mathbb{R} \to C, \quad f^\wedge(x, \lambda) = x' - \lambda \varphi(x(\cdot)),
\]

\[
f^\sim: C^1 \oplus \mathbb{R} \to C \oplus \mathbb{R}^n, \quad f^\sim(x, \lambda) = (f^\wedge(x, \lambda), x(0) - x(1)),
\]

\[
\Phi = Df(\xi, p), \quad \Phi^\wedge = Df^\wedge(\xi, p), \quad \Phi^\sim = Df^\sim(\xi, p),
\]

\[
\psi: \Omega \to C^1 \oplus \mathbb{R}, \quad \psi(w, \lambda) = (x, \lambda), \text{ where } x(t) = \eta(w, t\lambda),
\]

\[
A^\sim: C^1 \oplus \mathbb{R} \to \mathbb{R}^n, \quad A^\sim(x, p) = x(0) - x(1).
\]

It is easy to check that \( \Phi^\wedge(x, \lambda) = x' - pD\varphi(\xi(\cdot))x(\cdot) - \lambda \varphi(\xi(\cdot)) \). Note that \( f^\wedge(\psi(w, \lambda)) = 0 \) for all \((w, \lambda) \in \Omega \) and thus \( D\psi(v, p)(\mathbb{R}^n \oplus \mathbb{R}) \subset \text{Ker} \Phi^\wedge \). Let

\[
A: \text{Ker} \Phi^\wedge \to \mathbb{R}^n
\]

denote the restriction of \( A^\sim \) and

\[
\gamma: C^1 \cap \text{Ker} \Phi^\wedge \to \mathbb{R}^n, \quad \gamma(x) = x(0).
\]

Clearly \( \gamma \) is an isomorphism.

7.5. **Lemma.** With the above notations \( A(\text{Ker} \Phi^\wedge) = \mathbb{R}^n \) and \( \text{sgn}(P, \varphi(v)) = \text{sgn}_\lambda(A, \xi') \).

**Proof.** Clearly \( \pi = A^\sim \psi \) and thus \( P = AD\psi(v, p) \). Since \( P \) is surjective, \( A \) is also surjective. Let \( B: \mathbb{R}^n \to \text{Ker} \Phi^\wedge \) denote the restriction of \( D\psi(v, p) \).

Since \( \gamma(\psi(w, p)) = w \), \( B \) is injective. Moreover, \( B(\mathbb{R}^n) = C^1 \cap \text{Ker} \Phi^\wedge \). Thus, since \( D\psi(v, p)(0, 1) \neq C^1 \cap \text{Ker} \Phi^\wedge \) and \( \dim \text{Ker} \Phi^\wedge = n + 1 \), \( D\psi(v, p) \) maps \( \mathbb{R}^n \oplus \mathbb{R} \) isomorphically onto \( \text{Ker} \Phi^\wedge \). Note that \( P = AD\psi(v, p) \) and \( \gamma B \) is the identity in \( \mathbb{R}^n \) and \( B(\varphi(v)) = p^{-1} \xi' \). Therefore 7.5 follows from 6.6.

7.6. **Lemma.** With the above notations we have:

(a) \( \Phi^\sim \) is surjective and \( \text{Ker} \Phi^\sim \) is the one-dimensional subspace of \( C^1 \oplus \mathbb{R} \) spanned by \( \xi' \).

(b) \( \text{sgn}_\lambda(\Phi^\sim, \xi') = \text{sgn}(P, \varphi(v)) \).

**Proof.** First observe that \( \Phi^\sim(x, \lambda) = (\Phi^\wedge(x, \lambda), A^\sim(x, \lambda)) \). Since \( A^\sim \) maps \( \text{Ker} \Phi^\wedge \) onto \( \mathbb{R}^n \) and \( \Phi^\wedge \) is surjective, it follows that \( \Phi^\sim \) is surjective and the proof of (a) is complete.
To prove (b) note first that $\text{Ker}\Phi^\sim = \text{Ker}\Phi^\wedge \cap \text{Ker}\Lambda^\sim$. Define $\Psi: C^1 \to C \oplus R^n$ by $\Psi(x) = (\Phi^\wedge(x, 0), x(0))$. Obviously $\Psi$ is an isomorphism. Let $F = \Psi^{-1}(C)$, $F_0 = \Psi^{-1}(R^n)$. We have the direct sum decomposition

$$
\Psi = \begin{bmatrix}
\Gamma & 0 \\
0 & \gamma
\end{bmatrix}
\oplus \rightarrow \oplus,
\begin{array}
F \\
C \\
F_0 \\
R^n
\end{array}
$$

where $\Gamma: F \to C$ denotes the restriction of $\Psi$ and $\gamma(x) = x(0)$.

Recall that $\Lambda: \text{Ker}\Phi^\wedge \to R^n$ denotes the restriction of $\Lambda^\sim$. Since $F_0 \subset \text{Ker}\Phi^\wedge$, $C^1 \oplus R = F \oplus \text{Ker}\Phi^\wedge$. We have

$$
\Phi^\sim = \begin{bmatrix}
\Gamma & 0 \\
0 & \gamma_1
\end{bmatrix}
\oplus \rightarrow \oplus,
\begin{array}
F \\
C \\
F_0 \\
R^n
\end{array}
$$

where $\gamma_1(x) = x(1)$.

Using 6.5 we have $\text{sgn}_\Psi(\Phi^\sim, \xi^\prime) = \text{sgn}_r(\Gamma) \cdot \text{sgn}_s(\Lambda, \xi^\prime)$. If we let $S_\tau(x) = (x' + x, x(0) - \tau x(1))$, then for each $\tau \in [0, 1]$, $S_\tau \in \text{GL}_C(S)$. Thus, by 6.4(b), $\text{sgn}_s S_0 = 1$. Now define $\eta: [0, 1] \to \text{GL}_C(S)$ by

$$
\eta(\tau)(x) = ((1 - \tau)\Phi^\wedge(x, 0) + \tau(x' + x), x(0)).
$$

Since $\eta(0) = \Psi$ and $\eta(1) = S_0$ we have, using again 6.4(b), $\text{sgn}_r \Psi = 1$. Therefore, by 6.4(c),

$$
\text{sgn}_\Psi(\Phi^\sim, \xi^\prime) = \text{sgn}_s(\Phi^\sim, \xi^\prime)
$$

and the proof is finished.

7.7. Lemma. $\text{sgn}_s(\Phi^\sim, \xi^\prime) = \text{sgn}_r(\Phi, \xi^\prime)$.

Proof. Let $R: S^{-1}(C) \to C$ be the restriction of $S$, and $\Phi_1: S^{-1}(C) \oplus R \to C$ the restriction of $\Phi^\sim$. Consider the direct sum decompositions

$$
\Phi^\sim = \begin{bmatrix}
\Phi_1 & \Phi_0
\end{bmatrix}
S^{-1}(C) \oplus R \\
0 & S_0
C \\
R^n
$$

where $S_0$ denotes the restriction of $S$, and

$$
\Phi_1 = \begin{bmatrix}
\Phi & \Phi_2
\end{bmatrix}
R^{-1}(E) \oplus R \\
0 & R_0
E \\
E
$$

where $E_0 = \{x \in E: x(t) = tw, w \in R^n\}$ and $R_0$ denotes the restriction of $R$. We have $C = E \oplus E_0$. Applying 6.5 we obtain the desired result.
The proof of Proposition 7.4. The equality \( \text{sgn}_T(Df(ξ, p), ξ') = \text{sgn}(P, ϕ(v)) \) follows at once from 7.6 and 7.7.

The last step in our proof of 7.3 is the following generalization of 7.4.

**7.8. Proposition.** Suppose that \( v \in R^n \) is an elementary \( p \)-periodic point of multiplicity \( m \) with respect to equation (\( ∗ \)). Assume \( m = kv \) and let \( K = Z_k \). Let \( f^K: (E^1)^K \oplus R \to E^K \) denote the restriction of \( f \). Define \( ξ \in E^1 \) by \( ξ(t) = η(v, pt) \). Then \( Df^K(ξ, p) \) is surjective, \( ξ \in \text{Ker} Df^K(ξ, p) \) and

\[
\text{sgn}_T(Df^K(ξ, p), ξ') = \text{sgn}(P_k, ϕ(v)),
\]

where \( P_k \) denotes the linear Fuller map of (\( ∗ \)) at \( (v, k^{-1}p) \).

**Proof.** Consider the following isomorphisms of Banach spaces:

\[
\begin{align*}
Γ: (E^1)^K \oplus R &\to E^1 \oplus R, \quad Γ(x, λ) = (y, λ), \text{ where } y(t) = κx(k^{-1}t), \\
Ψ: E^K &\to E, \quad Ψ(x) = y, \text{ where } y(t) = x(k^{-1}t), \\
R: E^1 &\to E, \quad R(x) = x' + k^{-1}x.
\end{align*}
\]

Together with (\( ∗ \)) we consider the equation

\[
(α_*) \quad y'(t) = ψ(y(t)),
\]

where \( ψ: R^n \to R^n \) is defined by \( ψ(w) = φ(k^{-1}w) \).

Recall that we have assumed that, with respect to equation (\( ∗ \)), \( v \in R^n \) is a nonstationary elementary \( p \)-periodic point and \( (v, p) \) is of multiplicity \( m \). Furthermore, we have denoted by \( ξ \in C^1 \) the solution of the equation \( x'(t) = pφ(x(t)) \) satisfying the initial condition \( x(0) = v \). Define

\[
F: E^1 \oplus R \to E, \quad F(y, λ) = y' - λψ(y(·)),
\]

and let \( (ξ, p) = Γ(ξ, p) \). Since \( Ψf^K = FΓ \), we have \( ΨDf^K(ξ, p) = DF(ξ, p)Γ \).

Furthermore, \( Γ(ξ') = kξ' \). Therefore, applying 6.6, we have

\[
\text{sgn}_T(Df^K(ξ, p), ξ') = \text{sgn}_R(DF(ξ, p), ξ').
\]

Set \( ω(τ) = (1 - τ)T + τR \); clearly \( ω(τ) \) is an isomorphism for all \( τ \in [0, 1] \); hence, by 6.4, \( \text{sgn}_R R = 1 \) and thus \( \text{sgn}_R(DF(ξ, p), ξ') = \text{sgn}_T(DF(ξ, p), ξ') \). Thus

(i) \( \text{sgn}_T(Df^K(ξ, p), ξ') = \text{sgn}_T(DF(ξ, p), ξ') \).

Note that \( π^- \), the nonlinear Fuller map for (\( α_* \)), is given by

(ii) \( π^-(w, λ) = w - κη(k^{-1}w, k^{-1}λ) \).

Since \( v \) is an elementary \( p \)-periodic point for (\( ∗ \)), \( kv \) is one for (\( α_* \)). Let \( P^- \) denote the linear Fuller map for (\( α_* \)) at \((kv, p)\). From (ii) it follows at once that \( \text{sgn}(P^-, ϕ(kv)) = \text{sgn}(P_k, ϕ(v)) \). Therefore using (i) and 7.4 we obtain

\[
\text{sgn}_T(Df^K(ξ, p), ξ') = \text{sgn}(P_k, ϕ(v)).
\]
Proof of Theorem 7.3. The theorem follows at once from Proposition 7.8 and Theorem 6.7.

8. S\(^1\)-degree and the Fuller index. As in Section 7 we consider the autonomous differential equation

\[(*)\quad y'(t) = \varphi(y(t)),\]

where \(\varphi : \mathbb{R}^n \to \mathbb{R}^n\) is a \(C^\infty\) map. In what follows we keep the notations introduced at the beginning of the preceding section.

Suppose \(U\) is an open subset of \(\mathbb{R}^n \oplus \mathbb{R}\) such that \(U \subset \Omega_+\). We say that \(U\) is admissible for \(\varphi\) if \(\partial U \cap \Sigma = \emptyset\) (i.e. there is no periodic point on the boundary of \(U\)). Recall that if \(U\) is admissible for \(\varphi\) then there is defined a rational number \(i_p(\varphi, U)\), the Fuller index of \(\varphi\) with respect to \(U\) (cf. Fuller [11], Chow–Mallet-Paret–Yorke [5]). Assuming that \(U\) is admissible for \(\varphi\) let \(C = U \cap \Sigma\); clearly \(C\) is compact. Define \(\Phi : C \to E^1 \oplus \mathbb{R}\) by

\[\Phi(v, p) = (x, p),\quad \text{where } x(t) = \eta(u, tp).\]

Since \(\Phi\) is continuous, \(\Phi(C)\) is compact. Moreover, \(\Phi(C) \subset f^{-1}(0)\) and \(f^{-1}(0) \setminus \Phi(C)\) is closed in \(E^1 \oplus \mathbb{R}\). Therefore there exists an open bounded subset \(\mathcal{U} \subset E^1 \oplus \mathbb{R}\) such that \(f^{-1}(0) \cap \mathcal{U} = \Phi(C)\) and \(f^{-1}(0) \cap \partial \mathcal{U} = \emptyset\). Thus \(\text{Deg}(f, \mathcal{U}, T)\) is defined. For \(m \in \mathbb{N}\) set

\[d_m(\varphi, U) = \deg_H(f, \mathcal{U}, T),\quad \text{where } H = Z_m.\]

Now we are able to state the principal result of this section.

8.1. Theorem. If \(U \subset \Omega_+\) is admissible for \(\varphi\) then

\[i_p(\varphi, U) = \sum_{m \in \mathbb{N}} d_m(\varphi, U).\]

Proof. First, consider a special case: there is an elementary periodic point \((v, p) \in U\) such that

\[C = \{\eta(v, t) ; 0 \leq t \leq p\}.\]

In other words, we assume that \(C\) is one periodic orbit. Let \(m\) be the multiplicity of \((v, p)\) and \(m \cdot q = p\). Let \((as in 7.3)\) \(P_m\) denote the linear Fuller map of \((*)\) at \((v, q)\) and if \(m\) is even, \(m = 2\mu\), let \(P\) denote the linear Fuller map of \((*)\) at \((v, p)\). It follows from 7.3 that there are two possibilities:

(a) \(\text{sgn}(P, \varphi(v)) = \text{sgn}(P_m, \varphi(v))\);

in this case \(d_k(\varphi, U) = 0\) for \(k \neq m\), \(d_k(\varphi, U) = \text{sgn}(P, \varphi(v))\) for \(k = m\) and because from the definition of the Fuller index we have \(i_p(\varphi, U) = m^{-1}\text{sgn}(P, \varphi(v))\), the conclusion follows at once;

(b) \(m\) is even and \(\text{sgn}(P, \varphi(v)) = -\text{sgn}(P_m, \varphi(v))\);
in this case

$$i_F(\varphi, U) = m^{-1} \text{sgn}(P, \varphi(v)) = (\mu^{-1} - m^{-1}) \text{sgn}(P, \varphi(v))$$

$$= \mu^{-1} d_\mu(\varphi, U) + m^{-1} d_m(\varphi, U)$$

and the conclusion follows.

Thus we have proved our theorem in the case where \( \varphi \) has one elementary periodic orbit in \( U \). Therefore, using the additivity of the \( S^1 \)-degree and the definition of the Fuller index completes the proof.

Note that from Theorem 8.1 we see at once that the Fuller index may be defined in terms of the \( S^1 \)-degree. The crucial step made by Fuller in [11] was to prove the homotopy invariance of \( i_F(\varphi, U) \); using the \( S^1 \)-degree one can obtain an independent proof of this property.

References


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