Integration of infinite systems of differential inequalities

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In this paper we investigate infinite systems of ordinary differential inequalities of the form

$$D\varphi_i(t) \leq f_i(t, \varphi_1(t), \varphi_2(t), ..., \varphi_n(t), ...)$$

for any of the Dini derivatives of the function $\varphi(t)$ at the point $t$. The first section of the paper deals with some simple linear differential inequalities. The second section concerns the non-linear case. We there introduce the right-hand maximum solution of a countable system of differential equations.

1. For the sake of clarity we will consider, in the following, inequalities with right-hand upper derivatives. Remember that the right-hand upper derivative denoted by $\bar{D}_+\varphi(t)$ is defined by

$$\bar{D}_+\varphi(t) = \limsup_{h \to 0^+} \frac{\varphi(t+h) - \varphi(t)}{h}.$$ 

It can be remarked that our theorems remain true if $\bar{D}_+$ is replaced by any other Dini derivative.

We start with the following fundamental lemma:

**Lemma 1** (see [2]). Let $\Omega$ be an open subset of the $n$-dimensional space of points $(y_1, ..., y_n)$. Suppose that the functions $g_i(t, y_1, ..., y_n)$ $(i = 1, ..., n)$ are continuous on $(0, a) \times \Omega$, and satisfy there the condition:

(W) If $y_k \leq \overline{y}_k$ for $k \neq i$ then $g_i(t, y_1, ..., \overline{y}_{i-1}, y_i, \overline{y}_{i+1}, ..., \overline{y}_n)$

$$\leq g_i(t, \overline{y}_1, ..., \overline{y}_{i-1}, y_i, \overline{y}_{i+1}, ..., \overline{y}_n).$$

Let the continuous functions $\varphi_i(t), ..., \varphi_n(t)$ satisfy in $(0, a)$ the inequalities

$$\bar{D}_+\varphi_i(t) \leq g_i(t, \varphi_1(t), ..., \varphi_n(t)),$$  

for $i = 1, ..., n$.

Suppose that the right-hand maximum solution $\omega_1(t), ..., \omega_n(t)$ (1) of the

(1) For the definition and construction of the extremal solutions of finite systems of differential equations see [2].
system \( y'_i = g_i(t, y_1, \ldots, y_n) \) \((i = 1, \ldots, n)\) exists in \(0, a\) and \(\varphi_i(0) \leq \omega_i(0)\). Then \(\varphi_i(t) \leq \omega_i(t)\) for \(i = 1, \ldots, n\) and \(t \in (0, a)\).

**Corollary.** If \(\varphi_i(t)\) \((i = 1, \ldots, n)\) satisfy in \(0, a\) the inequalities \(\bar{D}_+ \varphi_i(t) \geq g_i(t, \varphi_1(t), \ldots, \varphi_n(t))\) \((i = 1, \ldots, n)\) then \(\varphi_i(t) \geq \tau_i(t)\) where \(\tau_i(t)\) are the components of the right-hand minimum solution of the system \(y'_i = g_i(t, y_1, \ldots, y_n)\) \((i = 1, \ldots, n)\) such that \(\varphi_i(0) \geq \tau_i(0)\).

Consider an infinite matrix \(\{a_{ik}\}, \ i, k = 1, 2, 3, \ldots\), with real constants \(a_{ik}\). Denote by \(\{u_{ik}(t)\}_{n}\) the fundamental matrix of the system of differential equations

\[
z'_i = \sum_{k=1}^{n} a_{ik} z_k, \quad i = 1, 2, \ldots, n.
\]

Notice that \(\sum_{k=1}^{n} u_{ik}(0) = 1\). We will prove the following lemma:

**Lemma 2.** Suppose the elements of \(\{a_{ik}\}\) satisfy \(a_{ik} \geq 0\) for \(i \neq k\). Assume that we are given a sequence of non-negative functions \(\{\psi_k(t)\}\) which are continuous on \(0, a\). Let the series \(\sum_{k=1}^{\infty} a_{ik} \psi_k(t)\) be convergent for every \(i = 1, 2, \ldots\) and every \(t \in (0, a)\). Suppose that

\[
(1) \quad \sum_{k=1}^{\infty} a_{ik} \psi_k(t) \leq \bar{D}_+ \psi_i(t), \quad i = 1, 2, \ldots, \quad t \in (0, a)
\]

and \(1 \leq \psi_i(0)\) for \(i = 1, 2, \ldots\) Under our assumptions we have

\[
\sum_{k=1}^{n} u_{ik}(t) \leq \psi_i(t)
\]

for arbitrary \(n\) and \(i, t \in (0, a)\).

**Proof.** It is a simple matter to verify that the functions \(u_i(t) = \sum_{k=1}^{n} u_{ik}(t)\), \(i = 1, 2, \ldots, n\), satisfy the system

\[
u_i'(t) = \sum_{k=1}^{n} a_{ik} u_k(t), \quad i = 1, 2, \ldots, n
\]

and \(u_i(0) = 1\). On the other hand \(a_{ik} \geq 0\) for \(i \neq k\) and \(\psi_k(t) \geq 0\) for \(i, k = 1, 2, \ldots\). It follows then from (1) that

\[
\sum_{k=1}^{n} a_{ik} \psi_k(t) \leq \bar{D}_+ \psi_i(t), \quad i = 1, 2, \ldots, n
\]
and consequently
\[ \sum_{k=1}^{n} a_{ik} [\psi_k(t) - u_i(t)] \leq D_+[\psi_i(t) - u_i(t)], \quad i = 1, \ldots, n. \]

But \( 0 \leq \psi_k(0) - u_k(0), \quad k = 1, \ldots, n. \) The assertion of the lemma follows from the last differential inequalities and from the above corollary.

Lemma 2 generalizes certain result of [1] (lemma 3, p. 249) where appeared the assumptions: \( a_{ik} \geq 0 \) for \( i \neq k \) and \( \sum_{k=1}^{\infty} a_{ik} = 0 \) for each \( i \).

However, the above equality shows that we can take in our lemma \( \psi_k(t) = 1. \)

**Theorem 1.** Let \( \{a_{ik}\} \) be an infinite matrix of real constants such that
\[ a_{ik} \geq 0 \quad \text{for} \quad i \neq k. \]

Suppose that there exists a sequence \( \{\psi_k(t)\} \) of non-negative functions which are continuous in \( (0, a) \) and satisfy
\[ \sum_{k=1}^{\infty} a_{ik} \psi_k(t) \leq D_+ \psi_i(t) \quad \text{for} \quad i = 1, 2, \ldots, 
\quad 0 < t < a, \]
\[ 1 \leq \psi_i(0) \quad \text{for} \quad i = 1, 2, \ldots. \]

Suppose we are given a sequence \( \{\varphi_i(t)\} \) of functions, which are continuous on \( (0, a) \). Assume that for every \( i \) the series \( \sum_{k=1}^{\infty} a_{ik} \varphi_k(t) \) is almost uniformly convergent on \( (0, a) \). Suppose that the functions \( \varphi_i(t) \) satisfy the following inequalities
\[ D_+ \varphi_i(t) \leq \sum_{k=1}^{\infty} a_{ik} \varphi_k(t), \quad i = 1, 2, \ldots, \quad t \in (0, a), \]
\[ \varphi_i(0) \leq 0 \quad \text{for} \quad i = 1, 2, \ldots. \]

Let us assume that
\[ \sigma_n(t) = \max_{k=1, \ldots, n} \left| \sum_{k=n+1}^{\infty} a_{kn} \varphi_n(t) \right| \to 0 \]
almost uniformly on \( (0, a) \).

Under our assumptions the inequalities
\[ \varphi_i(t) \leq 0 \]
hold for \( i = 1, 2, \ldots \) and \( t \in (0, a) \).

**Proof.** Given a fixed \( n \) denote by \( \lambda_i^n(t) = \sum_{k=n+1}^{\infty} a_{ik} \varphi_k(t) \). Take now the system
\[ y'_i = \sum_{k=1}^{n} a_{ik} y_k + \lambda_i^n(t), \quad i = 1, \ldots, n \]
and denote by $\gamma_i(t)$, ..., $\gamma_n(t)$ its solution such that $\gamma_i(0) = 0$ for $i = 1, ..., n$. Observe that (4) implies

$$
D_t \varphi_i(t) \leq \sum_{k=1}^{n} a_{ik} \varphi_k(t) + \lambda_i^n(t), \quad \varphi_i(0) \leq \gamma_i(0), \quad i = 1, ..., n.
$$

Applying (2) and lemma 1 we thus get that

$$
\varphi_i(t) \leq \gamma_i(t) \quad \text{for} \quad i = 1, ..., n, \quad 0 \leq t < a.
$$

On the other hand

$$
\gamma_i(t) = \int_0^t \sum_{k=1}^{n} u_{ik}(t-\tau) \lambda_i^n(\tau) d\tau.
$$

Hence

$$
\varphi_i(t) \leq \int_0^t \sum_{k=1}^{n} u_{ik}(t-\tau) \lambda_i^n(\tau) d\tau, \quad i = 1, ..., n.
$$

Notice now, that by the corollary $u_{ik}(t-\tau) \geq 0$ and obviously $\lambda_i^n(\tau) \leq \sigma^n(\tau)$. Therefore

$$
\varphi_i(t) \leq \int_0^t \sum_{k=1}^{n} u_{ik}(t-\tau) \sigma_n(\tau) d\tau.
$$

But (3) and lemma 2 imply that

$$
\sum_{k=1}^{n} u_{ik}(t-\tau) \leq \varphi_i(t-\tau), \quad i = 1, ..., n.
$$

We have then

$$
\varphi_i(t) \leq \int_0^t \varphi_i(t-\tau) \sigma_n(\tau) d\tau, \quad \text{for} \quad i = 1, ..., n.
$$

The limit passage in the above inequalities together with (5) proves the assertion.

2. This section deals with non-linear inequalities. We assume in the following that the functions $f_i(t, y_1, y_2, ..., y_n, ...) \ (i = 1, 2, ...)$ are defined for $t \in (0, a)$ and for arbitrary real-valued sequences $y = (y_k)$. They are continuous in the following sense: for each $i$, if for every $k$, $y_k^{(r)} \rightarrow y_k$ and $t' \rightarrow t$ then $f_i(t', y_1^{(r)}, y_2^{(r)}, ..., y_n^{(r)}, ...) \rightarrow f_i(t, y_1, y_2, ..., y_n, ...)$. The following condition generalizes the condition (W) of lemma 2:

(C) for every $i$, if $\bar{y}_k \leq \bar{y}_k$ for $k \neq i$ then

$$
f_i(t, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{i-1}, y_i, \bar{y}_{i+1}, ...) \leq f_i(t, \bar{y}_1, \bar{y}_2, ..., \bar{y}_{i-1}, y_i, \bar{y}_{i+1}, ...).
$$
We begin with the following

**Theorem 2.** Suppose that the functions \( f_i(t, y_1, y_2, \ldots, y_n, \ldots) \) \((i = 1, 2, \ldots)\) satisfy (C). We assume that there exist finite constants \( M_i > 0 \) such that

\[
|f_i(t, y_1, y_2, \ldots, y_n, \ldots)| \leq M_i, \quad i = 1, 2, \ldots
\]

for \( t \in \langle 0, a \rangle \) and \( y = \{ y_k \} \) arbitrary. Suppose we are given a sequence of functions \( \{ \varphi_i(t) \} \) which are continuous on \( \langle 0, a \rangle \) and satisfy on \( (0, a) \) the following inequalities

\[
\bar{D}_t \varphi_i(t) \leq f_i(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots).
\]

Then there exists the solution \( \{ \omega_i(t) \} \) in \( \langle 0, a \rangle \) of the infinite system

\[
\omega_i(t) = f_i(t, \omega_1(t), \omega_2(t), \ldots, \omega_n(t), \ldots), \quad i = 1, 2, \ldots
\]

such that \( \omega_i(0) = \varphi_i(0) \) and

\[
\varphi_i(t) \leq \omega_i(t) \quad \text{for} \quad t \in \langle 0, a \rangle \quad \text{and} \quad i = 1, 2, \ldots
\]

**Proof.** Suppose that the continuous functions \( \varphi_i(t) \) satisfy on \((0, a)\) the inequalities

\[
\bar{D}_t \varphi_i(t) \leq f_i(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots), \quad i = 1, 2, \ldots
\]

Let us consider the following differential equation

\[
y' = F_i(t, y) = f_i(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_{i-1}(t), y, \varphi_{i+1}(t), \ldots).
\]

Obviously \( F_i \) is continuous in \((t, y)\). On the other hand (10) implies \( \bar{D}_t \varphi_i(t) \leq F_i(t, \varphi_i(t)) \). Hence by lemma 1

\[
\varphi_i(t) \leq \psi_i^1(t) \quad \text{for} \quad t \in \langle 0, a \rangle
\]

where \( \psi_i^1(t) \) is the right-hand maximum solution of (11) such that \( \psi_i(0) = \psi_i^1(0) \). This maximum solution exists in the whole interval \( \langle 0, a \rangle \). This is an immediate consequence of the boundedness of \( f_i \). We have also

\[
\frac{d}{dt} \psi_i^1(t) = F_i(t, \varphi_i(t)).
\]

By (12), (13) and condition (C) we get

\[
\frac{d}{dt} \psi_i^1(t) \leq f_i(t, \varphi_1^1(t), \varphi_2^1(t), \ldots, \varphi_n^1(t), \ldots), \quad i = 1, 2, \ldots
\]

We see now that to every sequence \( \{ \psi_k(t) \} \) of functions which satisfy (10) there corresponds a sequence \( \{ \psi_k^1(t) \} \) such that the conditions (12), (13) and (14) hold. We have just to do with a transformation law which maps \( \psi = \{ \psi_k(t) \} \) on the sequence \( \psi' = \{ \psi_k^1(t) \} \). Denote this transformation by \( F \).
Hence $F^P = \psi^1$. It follows from (14) that we can apply $F$ to $\psi^i$, or more generally, that the sequence $\psi^{n+1} = F\psi^n$ is well defined. It is easy to prove that
\begin{equation}
\psi^n_i(t) \leq \psi^{n+1}_i(t) \tag{15}
\end{equation}
and
\begin{equation}
\frac{d}{dt} \psi^n_i(t) = f_i(t, \psi^{n-1}_i(t), \ldots, \psi^{n-1}_{i-1}(t), \psi^n_i(t), \psi^{n-1}_{i+1}(t), \ldots) \tag{16}
\end{equation}
The inequalities $|\psi^n_i(t)| \leq M_i + |\psi_i(0)|, \left| \frac{d}{dt} \psi^n_i(t) \right| \leq M_i$ show that for a fixed $i$ the sequence $\{\psi^n_i(t)\}$ is equibounded and equicontinuous on every compact contained in $<0, a)$. Hence, the limits $\lim_{n \to \infty} \psi^n_i(t) = \omega_i(t)$ exist and the convergence is almost uniform on $<0, a)$. It follows from (16) that $\omega'_i(t) = f_i(t, \omega_i(t), \omega_i(t), \ldots, \omega_i(t), \ldots)$. By (15) we conclude that $\psi_i(t) \leq \omega_i(t)$ for $t \in (0, a)$ and $i = 1, 2, \ldots$ Then assertion of our theorem follows if we put $\psi_i(t) = \varphi_i(t)$.

The supposed inequalities $|f_i| \leq M_i$ may be replaced by the following assumption: $|f_i(t, y_1, \ldots, y_n)| \leq g_i(|y_i|)$, $g_i(t, z)$ are continuous and for an arbitrary initial value $y_i(t) \geq 0$ the right-hand maximum solution of equation $y' = g_i(t, y)$ passing through the point $(0, y_i(0))$ exists in the whole interval $<0, a)$.

It is easy to see that the method used in the proof of theorem 2 does not need the assumption that the considered systems are countable. Hence, in theorem 2 the countable systems may be replaced by an arbitrary infinite systems. However, in the case of countable systems theorem 2 can be proved by using following arguments: suppose that the sequence $\{\varphi_i(t)\}$ satisfies (7) and consider the finite system
\begin{equation}
y^n_i = F^n_i(t, y_1, \ldots, y_n) = f_i(t, y_1, y_2, \ldots, y_n, \varphi_{n+1}(t), \varphi_{n+2}(t), \ldots), \tag{17}
i = 1, 2, \ldots, n.
\end{equation}
Condition (C) implies that $F^n_i$ satisfy contition (W) of lemma 1. Denote by $\omega^n_1(t), \ldots, \omega^n_n(t)$ the right-hand maximum solution of (17) such that $\omega^n_i(0) = \varphi_i(0)$. The inequalities (7) imply the following inequalities
\begin{equation}
D_i \varphi_i(t) \leq F^n_i(t, \varphi_1(t), \ldots, \varphi_n(t)), \quad i = 1, \ldots, n. \tag{18}
\end{equation}
Hence, by lemma 1
\begin{equation}
\varphi_i(t) \leq \omega^n_i(t) \quad \text{for} \quad i = 1, 2, \ldots, n, \quad t \in (0, a). \tag{18}
\end{equation}
Observe that by (C) and (18)
\begin{align*}
D_i \varphi_{n+1}(t) & \leq f_{n+1}(t, \omega_1(t), \ldots, \omega_n(t), \varphi_{n+1}(t), \varphi_{n+2}(t), \ldots) \\
& = F^n_{n+1}(t, \omega_1(t), \ldots, \omega_n(t), \varphi_{n+1}(t)).
\end{align*}
This inequality and the definition of \( \omega_i(t) \) imply by lemma 1

\[
\omega_i(t) \leq \omega_{i+1}(t), \quad i = 1, \ldots, n, \quad \varphi_{n+1}(t) \leq \omega_{n+1}(t).
\]

Arguments similar to those used in the proof of theorem 2 show that the limits \( \lim_{n \to \infty} \omega_i(t) \) are the components of a solution of system \( y'_i = f_i(t, y_1, \ldots, y_n, \ldots) \) and obviously \( \varphi_i(t) \leq \lim_{n \to \infty} \omega_i(t) \).

**Theorem 3.** Suppose that \( f_i \) satisfy (C) and \( |f_i(t) - f_i(s)| \leq M_i \lt +\infty \). Then for every sequence \( \hat{y} = \{ y_i \} \) there exists in \( <0, a) \) the right-hand maximum solution \( \{ \omega_i(t; \hat{y}) \} \) of the system

\[
y'_i = f_i(t, y_1, \ldots, y_n, \ldots), \quad i = 1, 2, \ldots, n
\]

such that \( \omega_i(0; \hat{y}) = \hat{y}_i \). If the functions \( \varphi_i(t) \) are continuous on \( <0, a) \) and satisfy on \( (0, a) \) the inequalities

\[
E_i \varphi_i(t) \leq f_i(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t), \ldots), \quad i = 1, 2, \ldots
\]

then \( \varphi_i(t) \leq \omega_i(t; \varphi(0)) \) \( \varphi(0) = \{ \varphi_i(0) \} \) for \( i = 1, 2, \ldots \) and \( t \in <0, a) \).

**Proof.** The functions \( \varphi_i(t) = \int_{0}^{t} f_i(s) \, ds \) satisfy (10). It follows from theorem 2 that there exists on \( <0, a) \) at least one solution \( \{ \omega_i(t) \} \) of (19) such that \( \omega_i(0) = \hat{y}_i \). Denote by \( \Omega_i \) the set of \( i \)-th components of solutions of (19) passing through \( (0, \hat{y}_i) \). We define now

\[
\omega_i(t; \hat{y}) = \sup_{\omega \in \Omega_i} \omega_i(t) \quad (\omega \in \Omega_i).
\]

The functions \( \omega \in \Omega_i \) are equibounded and equicontinuous in every compact subinterval of \( <0, a) \). We get therefore that \( \omega_i(t; \hat{y}) \) is continuous in \( t \) on \( <0, a) \). Let \( \{ \omega_i(t) \} \) be an arbitrary solution of (19) such that \( \omega_i(0) = \hat{y}_i \). We have

\[
\omega_i(t) = f_i(t, \omega_1(t), \ldots, \omega_n(t), \ldots), \quad i = 1, 2, \ldots
\]

and by (20)

\[
\omega'_i(t) \leq f_i(t, \omega_1(t; \hat{y}), \ldots, \omega_{i-1}(t; \hat{y}), \omega_i(t), \omega_{i+1}(t; \hat{y}), \ldots). \quad (21)
\]

It follows from (21) that

\[
\omega_i(t) \leq \sigma_i(t), \quad i = 1, 2, \ldots, \quad t \in <0, a) \quad (22)
\]
where $\sigma_i(t)$ is the right-hand maximum solution of the equation

$$y' = f_i(t, \omega_1(t: y), ..., \omega_{i-1}(t: y), y, \omega_{i+1}(t: y), ...)$$

such that $\sigma_i(0) = y_i$. The inequality (22) holds for an arbitrary solution. We get therefore

$$\omega_i(t: y) \leq \sigma_i(t)$$

and consequently by (20) and (C)

$$\sigma_i(t) \leq f_i(t, \sigma_1(t), ..., \sigma_n(t), ...).$$

By theorem 2 there exists a solution $\{\tau_i(t)\}$ of (19) such that $\tau_i(0) = y_i$ and

$$\sigma_i(t) \leq \tau_i(t).$$

But $\tau_i(t) \leq \omega_i(t: y)$ and by (24) and (25) we derive $\tau_i(t) = \omega_i(t: y)$. We have just proved that $\{\omega_i(t: y)\}$ is a solution of (19). It follows from (20) that this solution is the right-hand maximum one. The second part of the assertion follows easily from theorem 2 and from (20).

References


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