Applications of Denjoy analogue, I*  
(Sufficient conditions for a function to be monotone)  

by K. M. Garg ** (Lucknow)

In an earlier paper [1] the author has proved a theorem (1) called as "Denjoy analogue". In this note we investigate some applications of this theorem.

In 1951 T. Ważewski proved (2) that

(I) "If \( f(x) \) is a continuous function in an interval \( I \) and \( Q \) denotes the set of points \( x \in I \) for which

\[
D^+ f(x) < 0,
\]

a necessary and sufficient condition for \( f(x) \) to be monotone non-decreasing in \( I \) is that

\[
mf(Q) = 0.
\]

II) "If \( f(x) \) is a finite function of one variable such that (i) \( \limsup_{h \to 0^+} f(x-h) \leq f(x) \leq \limsup_{h \to 0^+} f(x+h) \) at every point \( x \), and (ii) \( D^+ f(x) \geq 0 \) at every point \( x \) except at most at those of an enumerable set, then the function \( f(x) \) is monotone non-decreasing."

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(1) For the sake of convenience this theorem is reproduced as follows:  
"Given an arbitrary finite real function \( f(x) \) defined in a linear interval \( I \), the Dini derivates of \( f(x) \) at each point \( x \), except possibly at a set \( A \subset I \) for which \( mf(A) = 0 \), satisfy one of the following four relations:

(a.1) \[ D^+ f(x) = D_+ f(x) = D_- f(x) = D_- f(x) \neq 0, \]

(a.2) \[ D^+ f(x) = + \infty, \quad D_- f(x) = - \infty, \quad D_- f(x) = D^- f(x) \neq 0, \]

(a.3) \[ D_+ f(x) = - \infty, \quad D^- f(x) = + \infty, \quad D^+ f(x) = D_- f(x) \neq 0, \]

(a.4) \[ D^+ f(x) = D^- f(x) = + \infty, \quad D_+ f(x) = D_- f(x) = - \infty. \]

(2) See Ważewski [5], p. 117, Theorem 1. The present enunciation follows from the theorem just on considering the function \(- f(x)\).
By an application of Denjoy analogue we prove (in § 1, Theorem 1) that the sufficiency part (*) of (I) remains valid even if (i) the continuity of \( f(x) \) is replaced by the condition (i) of (II), (ii) the measure in (**) is replaced by interior measure, and (iii) the set \( Q \) is replaced by one of its subsets. We also prove (in § 1, Theorem 2) that (II) remains still valid if the points where \( D^+ f(x) < 0 \) form a set of measure zero and if the points where \( D^+ f(x) = -\infty \) form a set whose power is less than that of the continuum.

The monotony of continuous functions which fulfil the Banach's conditions (*) (T_1) and (T_2) has been investigated in §§ 2 and 3.

1. Arbitrary real functions. Let \( f(x) \) be a finite real function of a real variable such that

\[
\limsup_{h \to 0^+} f(x - h) \leq f(x) \leq \limsup_{h \to 0^+} f(x + h)
\]

at every point \( x \), and let

\[
E = \{ x; D^+ f(x) \leq 0 \}.
\]

In case \( f(E) \) contains a non-degenerate interval, we evidently have

\[
m_{\text{f}}(E) > 0.
\]

But, according to the Denjoy analogue (see footnote (i)), we have

\[
m_{\text{f}}(E) = m_{\text{f}}(E_1 + E_2),
\]

where

\[
E_1 = \{ x; f'(x) \text{ exists (5) and is } < 0 \},
\]

and

\[
E_2 = \{ x; D^+ f(x) = D^- f(x) < 0 , \ D^+ f(x) = -\infty , \ D^- f(x) = +\infty \}.
\]

Thus, if \( m_{\text{f}}(E_1 + E_2) = 0 \), the set \( f(E) \) contains no non-degenerate interval, and so, according to a well known theorem (*) of A. Zygmund, \( f(x) \) is then monotone non-decreasing.

Hence we have the following

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(*) The necessity part remains trivially valid. In fact, if a function \( f(x) \), continuous or not, is monotone non-decreasing in \( I \), then it is known that all the four derivate of \( f(x) \) are \( > 0 \) throughout \( I \). (See Natanson [2], p. 208, Lemma 1.)

(**) For the definitions of Banach's conditions (T_1) and (T_2) see Saks [4], p. 277.

(*) The derivate \( f'(x) \) exists at a point \( x \) whenever the four derivate of \( f(x) \) are equal at \( x \), whether finite or infinite.

(*) Viz. "If \( f(x) \) is a finite function of a real variable such that (i) \( \limsup_{h \to 0^+} f(x - h) < f(x) \leq \limsup_{h \to 0^+} f(x + h) \) at every point \( x \), and (ii) the set of the values assumed by \( f(x) \) at the points \( x \) where \( D^+ f(x) \leq 0 \) contains no non-degenerate interval, then the function \( f(x) \) is monotone non-decreasing." (See Saks [4], p. 203, Theorem (7.1).)
THEOREM 1. If \( f(x) \) is a finite real function of a real variable such that

(i) \( \limsup_{h \to 0^+} f(x - h) \leq f(x) \leq \limsup_{h \to 0^+} f(x + h) \) at every point \( x \), and

(ii) the values assumed by \( f(x) \) at the points \( x \) where either \( f'(x) \) exists and is \( < 0 \), or

\[
D^+ f(x) = D_+ f(x) < 0, \quad D^- f(x) = -\infty, \quad D^- f(x) = +\infty,
\]

form a set whose interior measure is zero, then the function \( f(x) \) is monotone non-decreasing.

Let, again, \( f(x) \) be a finite real function which satisfies the relation (1) at every point \( x \), and let \( E_1 \) and \( E_2 \) be the sets as defined in (5) and (6).

Let, for \( i = 1 \) and \( 2 \), \( E_{i\infty} \) denote the set of those points of \( E_i \) where all the four derivatives of \( f(x) \) are infinite.

Since at any point \( x \in (E_1 - E_{1\infty}) \), or \( (E_2 - E_{2\infty}) \), \( f(x) \) has at least one derivate finite, it follows from a well known theorem (see [4], p. 271, Theorem (4.6)) of S. Saks that \( f(x) \) fulfils the Lusin’s condition (N) \(^7\) on either of the sets \( (E_1 - E_{1\infty}) \) and \( (E_2 - E_{2\infty}) \).

Hence, in case \( m(E_1 + E_2) = 0 \), we have

\[
m_f(E_1 - E_{1\infty}) = 0 = m_f(E_2 - E_{2\infty}),
\]

so that

\[
m_f(E_1 + E_2) = m_f(E_{1\infty} + E_{2\infty}). \tag{7}
\]

If \( m_f(E_{1\infty} + E_{2\infty}) > 0 \), the image-set \( f(E_{1\infty} + E_{2\infty}) \) has the power \( c \) \(^8\) and so the set \( (E_{1\infty} + E_{2\infty}) \) then also has the power \( c \).

Hence, in case the set \( (E_{1\infty} + E_{2\infty}) \) has a power less than \( c \), we have

\[
m_f(E_{1\infty} + E_{2\infty}) = 0. \tag{8}
\]

Combining the equations (7) and (8) it follows that, in case \( m(E_1 + E_2) = 0 \) and the set \( (E_{1\infty} + E_{2\infty}) \) has a power less than \( c \), we have

\[
m_f(E_1 + E_2) = 0,
\]

which in turn implies with the help of Theorem 1 that the function \( f(x) \) is then monotone non-decreasing.

We have thus proved the following

THEOREM 2. If \( f(x) \) is a finite real function of a real variable such that

(i) \( \limsup_{h \to 0^+} f(x - h) \leq f(x) \leq \limsup_{h \to 0^+} f(x + h) \) at every point \( x \).

\(^7\) For the definition of Lusin’s condition (N) see Saks [4], p. 224.

\(^8\) For, as the interior measure of a set, \( E \) is the least upper bound of the measures of all closed sets contained in \( E \), in case \( m_f E > 0 \), \( E \) contains a closed set \( F \) which has its measure \( > 0 \). Clearly, \( F \) is then an unenumerable closed set, and so has the power \( c \).

(See Natanson [2], p. 53.) The set \( E \) then evidently has the power \( c \).
(ii) the points where \( f'(x) \) exists and is \( < 0 \), or

\[
D^+ f(x) = D_+ f(x) < 0, \quad D_+ f(x) = -\infty, \quad D^- f(x) = +\infty,
\]
form a set of measure zero, and

(iii) the points where \( f'(x) \) exists and is \( = -\infty \), or

\[
D^+ f(x) = D_- f(x) = -\infty, \quad D_+ f(x) = -\infty, \quad D^- f(x) = +\infty,
\]
form a set whose power is less than that of the continuum, then the function \( f(x) \) is monotone non-decreasing.

2. Functions fulfilling Banach's condition \( (T_1) \). We first observe that the condition (i) of Theorem 1 is automatically satisfied in case the function \( f(x) \) is continuous.

Moreover, if a continuous function \( f(x) \) also fulfils the Banach's condition \( (T_1) \), the values assumed by \( f(x) \) at the points \( x \) where it has no derivative (finite or infinite) form a set of measure zero. (See [3], p. 130, Theorem 1, or [4], p. 278, Theorem (6.2)). Hence, in this case Theorem 1 gives

**Theorem 3.** Let \( f(x) \) be a continuous function which fulfils the Banach's condition \( (T_1) \). If the values assumed by \( f(x) \) at the points \( x \) where \( f'(x) \) exists and is \( < 0 \) form a set whose interior measure is zero, then the function \( f(x) \) is monotone non-decreasing.

Let, again, \( f(x) \) be a continuous function which fulfils the condition \( (T_1) \), and let

\[ E_1 = \{x; f'(x) \text{ exists and is } < 0\}, \]
\[ E_{1\infty} = \{x; f'(x) \text{ exists and is } = -\infty\}. \]

We have already observed in the proof of Theorem 2 that in case \( mE_1 = 0 \), and the set \( E_{1\infty} \) has a power \( < \infty \), we have

\[ m f(E_1) = 0. \]

But, according to the above Theorem 3, the last equation implies that the function \( f(x) \) is monotone non-decreasing.

Hence, we have the following

**Theorem 4.** Let \( f(x) \) be a continuous function which fulfils the Banach's condition \( (T_1) \). If (i) the points where \( f'(x) \) exists and is \( < 0 \) form a set of measure zero, and if (ii) the points where \( f'(x) = -\infty \) form a set whose power is \( < \infty \), then the function \( f(x) \) is monotone non-decreasing.

As a continuous function of bounded variation always fulfils the condition \( (T_1) \) (see [4], p. 279, Theorem (6.3)), the above Theorems 3 and 4 also hold for functions of bounded variation.
3. Functions fulfilling Banach's condition \((T_2)\). In case of a continuous function which fulfils the Banach's condition \((T_2)\), we have a slightly weaker result than that of Theorem 3, viz.:

**Theorem 5.** Let \(f(x)\) be a continuous function which fulfils the Banach's condition \((T_2)\). If the values assumed by \(f(x)\) at the points \(x\) where \(f'(x)\) exists and is \(< 0\) form a set whose measure is zero, then the function \(f(x)\) is monotone non-decreasing.

**Proof.** S. Saks proved (see [3], p. 133, Theorem 5 or [4], p. 280, Theorem (6.6)) in 1931 that

"If a continuous function \(f(x)\) fulfils the condition \((T_2)\) in an interval \([a, b]\), then
\[
-m_c f(N) \leq f(b) - f(a) \leq m_c f(P),
\]
where \(P\) and \(N\) denote respectively the sets of points of the interval \([a, b]\) where \(f(x)\) possesses a unique non-negative, non-positive, derivative."

Let \(f(x)\) be a continuous function which fulfils the condition \((T_2)\) in an interval \(I\).

Let, if possible, \(a\) and \(b\) be two points of \(I\) for which
\[
a < b, \quad f(a) > f(b).
\]

Since the function \(f(x)\) is continuous and fulfils the Condition \((T_2)\) in \([a, b]\), it follows from the above theorem of Saks that
\[
-m_c f(N \cdot [a, b]) \leq f(b) - f(a),
\]
where \(N\) denotes the set of points in \(I\) where \(f(x)\) has a non-positive derivative.

Denoting by \(E_1\) the set of points in \(I\) where \(f(x)\) has a derivative \(< 0\), we evidently have
\[
f'(x) = 0 \quad \text{for} \quad x \in N - E_1.
\]
This implies, according to another well known theorem \((9)\) of Saks, that
\[
m_f(N - E_1) = 0.
\]

Thus, in case there exist points \(a, b\) in \(I\) for which \((9)\) holds, we have, with the help of \((10)\) and \((12)\),
\[
-m_c f(E_1 \cdot [a, b]) \leq f(b) - f(a) < 0,
\]
i.e.
\[
m_c f(E_1 \cdot [a, b]) > 0.
\]

Hence, if we have \(m_c f(E_1) = 0\), there exist no points \(a, b\) in \(I\) for which the relation \((9)\) holds, and so the function \(f(x)\) is then monotone non-decreasing.

(*) Viz., "If one of the four Dini derivates of a function \(f(x)\) vanishes at every point of a set \(E\), then \(m_f(E) = 0\". See Saks [4], p. 272.
This completes the proof of Theorem 5.

In case a continuous function \( f(x) \) fulfils the \textit{Lusin's condition} (N), it also fulfils the condition (T\textsubscript{2}) (see [4], p. 284, Theorem (7.3)), and since in this case
\[
mf(E) = 0 \quad \text{whenever} \quad mE = 0,
\]
it follows from Theorem 5.

**Corollary** (\textsuperscript{19}). Let \( f(x) \) be a continuous function which fulfils the \textit{Lusin's condition} (N). If the points where \( f'(x) \) exists and is \( < 0 \) form a set of measure zero, then the function \( f(x) \) is monotone non-decreasing.

Let, once again, \( f(x) \) be a continuous function which fulfils the condition (T\textsubscript{2}). Let, as before,
\[
E_1 = \{x; f'(x) \text{ exists and is } < 0\}
\]
and
\[
E_{1\infty} = \{x; f'(x) \text{ exists and is } = -\infty\}.
\]
Since \( f'(x) \) is finite at each point of \( (E_1 - E_{1\infty}) \), the function \( f(x) \) fulfils the condition (N) on \( (E_1 - E_{1\infty}) \) (see [4], p. 271, Theorem (4.6)).

Hence, in case \( mE_1 = 0 \), we have
\[
mf(E_1 - E_{1\infty}) = 0,
\]
i.e.
\[
m_{\text{ef}}(E_1) = m_{\text{ef}}(E_{1\infty}).
\]
In case \( m_{\text{ef}}(E_{1\infty}) > 0 \), the image-set \( f(E_{1\infty}) \) is unenumerable, and so the set \( E_{1\infty} \) is also unenumerable.

Thus, in case \( mE_1 = 0 \) and the set \( E_{1\infty} \) is at most enumerable, we have
\[
m_{\text{ef}}(E_1) = m_{\text{ef}}(E_{1\infty}) = 0,
\]
which implies with the help of Theorem 5 that \( f(x) \) is then monotone non-decreasing.

The following result has thus been proved:

**Theorem 6.** Let \( f(x) \) be a continuous function which fulfils the Banach's condition (T\textsubscript{2}). If (i) the points where \( f'(x) \) exists and is \( < 0 \) form a set of measure zero, and if (ii) the points where \( f'(x) = -\infty \) are at most enumerable, then the function \( f(x) \) is monotone non-decreasing.

\textsuperscript{19} This corollary strengthens the following theorem of T. Ważewski (see [5], p. 118, Theorem 2 and the following remark 2): "If a function \( f(x) \), continuous in an interval \( I \), satisfies in this interval the Lusin's condition (N), and if the inequality \( D^*f(x) \geq 0 \) holds almost everywhere in \( I \), then the function \( f(x) \) is non-decreasing in the interval." 

The above corollary is, however, not new. (See Saks [4], p. 286.) This corollary is also known to hold in case the function \( f(x) \), instead of fulfilling the condition (N), possesses a derivative (finite or infinite) with the exception of points of an at most enumerable set. (See. Z. Zahorski [8], p. 19, Theorem 2.)
References


DEPARTMENT OF MATHEMATICS & ASTRONOMY,
UNIVERSITY OF LUCKNOW, INDIA.

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