A note on the coefficients of univalent functions

by J. T. Poole (Tallahassee, Fla.)

Let \( f \in \mathcal{S} \), i.e., let \( f \) be regular and univalent in \( |z| < 1 \) and have the series expansion

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n;
\]

for integral \( t, -\infty < t < \infty \), let

\[
f(z)^t = z^t + \sum_{n=t+1}^{\infty} a_n^{(0)} z^n.
\]

It is well known ([3]) that if \( f \) is extremal for the problem \( \max_{f \in \mathcal{S}} |a_n| \), then

\[
(n + 1) a_{n+1} = 2a_n a_{n-1} + (n-1) \overline{a}_{n-1}.
\]

(1)

The following interesting generalization of (1) is easily obtained.

**Theorem.** If \( f \in \mathcal{S} \) is extremal for the problem \( \max_{f \in \mathcal{S}} |a_n^{(0)}| \), then

\[
(n + 1) a_{n+1}^{(0)} = t(2a_n a_{n-1} + a_n^{(t-1)}) + (n-1) \overline{a}_{n-1}^{(0)}.
\]

(2)

**Proof.** For \( f \in \mathcal{S} \) we make the variation \( w^* = w + \varepsilon e^{i\theta} \), \( \varepsilon > 0 \), \( 0 \leq \theta \leq 2\pi \), of the image domain under \( f \). Then for small \( \varepsilon \)

\[
f^*(z) = f(z) + \varepsilon e^{i\theta} \left\{ 2a_n f(z) + 1 - zf'(z) \left( \frac{1}{z} - \bar{z} \right) \right\} + O(\varepsilon^2)
\]

belongs to \( \mathcal{S} \) ([5]). We use (3) to obtain a variational formula for the \( t \)th power of \( f \in \mathcal{S} \). Let

\[
k(z) = \left\{ 2a_n f(z) + 1 - zf'(z) \left( \frac{1}{z} - \bar{z} \right) \right\},
\]

then

\[
f^*(z)^t = \left[ f(z) + \varepsilon e^{i\theta} k(z) + O(\varepsilon^2) \right]^t
\]

\[
= f(z)^t + \varepsilon e^{i\theta} t f(z)^{t-1} k(z) + O(\varepsilon^2)
\]

\[
= f(z)^t + \varepsilon e^{i\theta} \left\{ 2t a_n f(z)^t + t f(z)^{t-1} - z f(z)^{t-1} \left( \frac{1}{z} - \bar{z} \right) \right\} + O(\varepsilon^2).
\]

(4)
Instead of attacking the problem \( \max_{f \in S} |a_n^{(0)}|, \ n = t+1, \ldots \), we may assume \( a_n^{(0)} \) is real ([5]) and thus consider the problem

\[
(5) \quad \max_{f \in S} \text{Re}(a_n^{(0)}), \quad n = t+1, \ldots
\]

Thus suppose \( f \in S \) is extremal for (5); by comparing coefficients in (4) and taking real parts we see that

\[
\text{Re} \left\{ \frac{a_n^{(0)}}{a_n^{(0)}} \right\} = \text{Re} \left\{ \varepsilon e^{i\theta} (2ta_n a_n^{(0)} + t_{n-1}^{(0)} - (n+1)a_{n+1}^{(0)} + (n-1)a_{n-1}^{(0)}) + O(\varepsilon^3) \right\} \leq 0
\]

for all values of \( \theta \in [0, 2\pi] \). This implies (2) and thus completes the proof of the theorem.

For \( f \in S \) let \( G = f (|z| < 1) \). Suppose \( D = \{ w \mid w = 1/\zeta, \ \zeta \in G \} \) and let \( E \) be the complement of \( D \) in the \( w \)-plane. Define the moments

\[
s_n = \int w^n \, d\mu, \quad n = 1, 2, \ldots,
\]

where \( \mu \) is the natural mass distribution on \( E \). Let

\[
\tilde{f}(z) = 1/f(1/z) = z + \sum_{n=0}^{\infty} a_n z^{-n}
\]

(\( \tilde{f} \) is regular, univalent and nonzero in \( |z| > 1 \), i.e., \( f \in \Sigma \)) and let

\[
\varphi(w) = w + \sum_{n=2}^{\infty} b_n w^n \quad \text{and} \quad \tilde{\varphi}(w) = w + \sum_{n=0}^{\infty} \tilde{b}_n w^{-n}
\]

be the inverses of \( f \) and \( \tilde{f} \) respectively. It is known ([1], [4]) that

1. \( \tilde{a}_n = a_n^{(-1)} \), \quad \( n = 0, 1, \ldots \),
2. \( b_n = \frac{1}{n} a_n^{(-n)} \), \quad \( n = 1, 2, \ldots \),
3. \( s_n = a_n^{(-n)} \), \quad \( n = 1, 2, \ldots \),
4. \( \tilde{b}_n = -\frac{1}{n} a_n^{(-n)} \), \quad \( n = 1, 2, \ldots \)

Therefore, making the appropriate substitution in (2), we have the following results.

I. If \( \tilde{f} \in \Sigma \) is extremal for the problem \( \max |\tilde{a}_n| \), then

\[
(n+1)\tilde{a}_{n+1} = 2\tilde{a}_0 \tilde{a}_n - \sum_{r=1}^{n+1} \tilde{a}_r \tilde{a}_{n-r} + (n-1)\tilde{a}_{n-1}, \quad \tilde{a}_{-1} = 1.
\]
II. If $\varphi \in S^{-1}$ is extremal for the problem $\max |b_n|$, then

$$(n+1)b_{n-1} = 2nb_n b_n - \sum_{r=1}^{n-1} \delta_r \delta_{n-r}, \quad b_1 = 1.$$ 

III. If $f \in S$ is extremal for the problem $\max |s_n|$, then

$$a_1^{(-n)} = n(2s_n s_n - s_{n+1}) - a_1^{(-n)}.$$ 

IV. If $\tilde{\varphi} \in \Sigma^{-1}$ is extremal for the problem $\max |\tilde{b}_n|$, then

$$(n+1)\tilde{b}_{n+1} = 2n\tilde{b}_n \tilde{b}_n - \sum_{r=1}^{n+1} \tilde{\delta}_r \tilde{\delta}_{n-r}, \quad \tilde{b}_{-1} = 1.$$ 

In obtaining II and IV we have used this fact ([2]) that if $\varphi$ is the inverse of $f$ and $\varphi(w)^t = \sum_{-1}^{\infty} b_v^{(0)} w^t$, then $b_v^{(0)} = \frac{t}{v} a_v^{(-v)}$, $v \neq 0$.

References


Reçu par la Rédaction le 5. 1. 1967