CERTAIN SUBDIRECT SUMS OF FINITE PRIME FIELDS

BY

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The fundamental notions used in this paper can be found in Jacobson [6], Kaplansky [8] and McCoy [12]. All rings considered here will be associative. For arbitrary subsets $B$ and $C$ of a ring $A$ the product $BC$ will mean the additive subgroup generated by all elements $bc$ with $b \in B$ and $c \in C$. The ring of rational integers will be denoted by $I$. For any element $a$ of the ring $A$, $Ia$ is the cyclic subgroup generated by $a$. Following Kandô [7], a ring $A$ is called strongly regular if $a \in a^2 A$ for any $a \in A$. Some characterizations of strongly regular rings have been given by Forsythe and McCoy [4], Kovács [9], Lajos — author [11] and author [16] (cf. also [17]). In part II of [11] it is shown that a ring is strongly regular if and only if its multiplicative semigroup is a semilattice of groups. Semigroups which are semilattices of groups (for their definition see Clifford [2]) were characterized also by Lajos [10].

The Boolean rings in which $a^2 = a$ holds for any element $a$ of the ring as well as the discrete direct sums of division rings are important instances of strongly regular rings. Any strongly regular rings is a subdirect sum of division rings [4]. On the other hand, the ring $I$ is a subdirect sum of the prime fields $I/(p)$, where $p$ runs over the set of all prime numbers, but $I$ is not strongly regular. We shall call ring $A$ a restricted Boolean ring (or an MPR-ring, respectively) if $a^2 = a$ and $ab = ba = a$, or $b$, or $0$ for any $a, b \in A$ (or if $A$ satisfies the minimum condition on principal right ideals of $A$, respectively; MPR-ring was in German denoted as "MHR-Ring", cf. [15]). As was shown by Gerčikov [5], a ring is a direct discrete sum of division rings if and only if it is an MPR-ring without non-zero nilpotent elements. Furthermore, by Satz 2.5 of part II (page 422) of [15], an MPR-ring $A$ has no non-zero nilpotent elements if and only if any right ideal $R$, contained in a principal right ideal $(a)_r = Ia + aA$ of $A$, contains a right unity element of $R$. Therefore Satz 2.5 of [15] yields also a characterization of discrete direct sums of division rings.
The aim of this paper is to characterize certain strongly regular subdirect sums of finite prime fields.

**Theorem 1.** For a ring $A$ the following two conditions are equivalent:

(I) any additive subgroup $S$ of $A$ is multiplicatively idempotent.

(II) $A$ is a direct sum of its ideals $A_2$ and $A_p$, i.e. $A = A_2 \oplus \sum_p A_p$,

where $A_2$ is a restricted Boolean ring, $p$ runs over the set of all different odd primes, and either $A_p \cong \mathbb{I}/(p)$ or $A_p = 0$.

**Corollary 2.** Any ring with condition (I) is a subdirect sum of finite prime fields.

**Corollary 3.** A ring $A$ without non-zero elements of odd additive order satisfies condition (I) if and only if it is a restricted Boolean ring.

**Corollary 4.** A ring $A$ without non-zero elements of even additive order satisfies condition (I) if and only if it is a torsion ring such that any non-zero $p$-component $A_p$ of $A$ is isomorphic to $\mathbb{I}/(p)$ (where $p \neq 2$).

**Proof of Theorem 1.** Assume that $A$ is a ring satisfying condition (I). Since the cyclic group $Ia$ is idempotent for any $a \in A$, there exists a number $m \in I$ such that $a = ma^2$. It can be noted that $e = ma$ is by

$$e^2 = m^2a^2 = m \cdot ma^2 = ma = e$$

idempotent. Furthermore, by

$$a = m^2a^2 = a^2 \cdot m^2a \in A_2$$

for any $a \in A$, $A$ is strongly regular and so it has no non-zero nilpotent elements.

We shall show that any element of $A$ has a square free additive order, that is, the additive group $A^+$ is elementary (cf. Kaplansky [8]). Namely, if $a = ma^2 \neq 0$, then $a^2 \neq 0$. Let $p$ be a prime number which does not divide the number $m$. Then by condition (I) there exists a number $n \in I$ such that $pa = n(pa)^2$, whence

$$(m-pn)pa^2 = pma^2 - npa = pa - pa = 0.$$ 

This means, by $a^2 \neq 0$, $p \neq 0$ and $m \neq pn$, that $A^+$ is not torsion free. If $T$ is the maximal torsion ideal of $A$, then the torsion free ring $A/T$ also satisfies condition (I); consequently, we have $A/T = 0$ and $T = A$. Let now, for an arbitrary prime number $p$, $A_p$ be a $p$-component of $A$. Then $A_p^2 = A_p$ and $(pA_p)^2 = pA_p$ imply $pA_p = p^2A_p$. Hence $pA_p^2$ is a divisible abelian group, which is, by $T = A$, a direct sum of Prüfer quasicyclic groups $C(p^\infty)$. Obviously, any $C(p^\infty)$ admits only trivial multiplication upon itself (i.e., $xy = 0$ for any $x, y \in C(p^\infty)$), contrary to condition (I). Consequently, for any $p$, $pA_p = 0$ (cf. I. Kaplansky [8]).

Since $A$ has no non-zero nilpotent elements, any idempotent belongs, according to a result of Forsythe and McCoy [4], to the centre $C$ of $A$. 
Therefore
\[ ma = e \epsilon C \quad \text{for } a = ma^2 \epsilon A. \]

Consequently, \( a^2 = a \) and \( b^2 = b \) imply \( ab = ba \).

We shall show that if \( p \) is an odd prime number, then \( A_p \not= 0 \) implies \( A_p \cong I/(p) \). For assume the existence of non-zero elements \( a \) and \( b \) with \( Ia \cap Ib = 0 \). Then \( a \) and \( b \) can be chosen, by condition (I), such that \( a^2 = a \) and \( b^2 = b \). Since \( S = Ia + Ib = S^2 \) is a subring, we have \( ab = ba = ka + lb \) with \( k, l \in I \). Now \( Ia \cap Ib = 0 \), \( a^2 b = ab \) and \( ab^2 = ab \) yield
\[ k^2 \equiv k, \quad l^2 \equiv l \quad \text{and} \quad kl \equiv 0 \pmod{p}; \]
consequently, \( ab = 0 \), or \( a \), or \( b \).

If \( ab = ba = 0 \), then there exists a number \( s \in I \) such that
\[ a + 2b = s(a + 2b)^2 = sa + 4sb, \]
whence, by \( Ia \cap Ib = 0 \),
\[ s \equiv 1, \quad 4s \equiv 2 \quad \text{and} \quad 4 \equiv 2 \pmod{p}, \]
and so we get \( p = 2 \) in a contradiction with the assumption \( p \not= 2 \). Similarly, if \( ab = ba = a \), then there exists a number \( t \in I \) such that
\[ a - 2b = t(a - 2b)^2 = t(-3a + 4b) \]
whence, by \( Ia \cap Ib = 0 \),
\[ 3t \equiv -1, \quad 4t \equiv -2, \quad t \equiv -1 \quad \text{and} \quad -3 \equiv -1 \pmod{p}, \]
and so we get the same contradiction \( p = 2 \) with the assumption \( p \not= 2 \).

The case \( Ia \cap Ib = 0 \), \( ab = ba = b \) is similarly impossible.

Therefore we have \( A_p \cong I/(p) \) for \( A_p \not= 0 \) and \( p \not= 2 \).

It is now sufficient to prove that any ring \( A \) with condition (I) and with an additive elementary 2-group is a restricted Boolean ring. In fact, condition (I) implies \( a^2 = a \) for any \( a \in A \) and
\[ ab = ba \epsilon Ia + Ib \]
for any \( a \) and \( b \) of \( A \).

Equality \( ab = a + b \) cannot occur for \( a \not= 0 \). Indeed, assuming \( ab = a + b \), the equations
\[ ab = a(ab) = a(a + b) = a + ab = a + a + b = b \]
would yield the contradiction \( a = 0 \) with the assumption \( a \not= 0 \). But \( a^2 = a \) and \( ab = ba \not= a + b \) for any \( a, b \in A \) mean that \( A = A_2 \) is a restricted Boolean ring.

Hence the implication (I) \( \Rightarrow \) (II) holds.

Conversely, assume that \( A \) is a ring with condition (II). Let \( S \) be an arbitrary additive subgroup of \( A \). According to condition (II), \( S \) has
an additive direct decomposition \( S = S_p + \sum_{p} S_p \), where \( p \) is an odd prime number. Consequently, \( S_p S_q = S_{pq} S_p = 0 \) for \( p \neq q \). Since, for any non-zero \( p \)-component \( S_p \), \( S_p \cong I/(p) \), there must be \( S_p^2 = S_p \), whence also
\[
\left( \sum_{p} S_p \right)^2 = \sum S_p.
\]

Since \( a^2 = a \) for any \( a \in A_2 \), we infer, by the definition of the product BC of subsets \( B \) and \( C \) of \( A \), that the 2-component \( S_2 \) of \( S \) satisfies \( S_2^2 \cong S_2 \). On the other hand, also \( S_2^2 \subseteq S_2 \) holds, because the 2-component \( A_2 \) of \( A \) is a restricted Boolean ring. Therefore \( S_2^2 = S_2 \), whence \( S^2 = S \).

Consequently, we have also the implication \((\text{II}) \Rightarrow (\text{I})\), which completes the proof.

Examples 5. (1) Let \( A \) be the algebra over the field of two elements, generated by the elements \( a, b \) and \( c \) with the table of multiplication

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Then \( A \) is a Boolean ring having eight elements such that the subgroup \( S = Ia + Ic \) is an idempotent subring, but the subgroup \( T = Ia + Jb \), satisfying \( T^2 = A \neq T \), is not a subring and is not idempotent. Therefore \( A \) is a Boolean (but not restricted Boolean) ring without condition \((\text{I})\).

(2) Let \( A \) be the complete direct sum of the fields \( K_{2,n} \) of two elements, \( n = 1, 2, 3, \ldots \) Furthermore, let \( a_n \) be the infinite vector, treated as an element in \( A \), which has 0 in the first \( n \) components and 1 elsewhere. Let \( b_n \) denote the product \( a_1 a_2 \ldots a_n \) of \( A \). Then \( A \) is a (restricted) Boolean ring, which is also strongly regular, but the infinite proper descending chain of principal ideals
\[
(b_1) \supset (b_2) \supset (b_3) \supset \ldots
\]
says that \( A \) is not an MPR-ring. Obviously, \( b_n \) is the unity element of the ideal \( (b_n) \). Let \( C_n \) be an ideal of \( A \) such that the direct decomposition
\[
(b_{n-1}) = (b_n) \oplus C_n
\]
holds for any \( n \geq 2 \). Construct the direct sum \( D = \bigoplus_{n \geq 2} C_n \). The ideal \( D \) of \( A \) lies in the principal ideal \( (b_1) \) of the (commutative) ring \( A \) of cardinality continuum, and the ring \( D \) does not contain unity element (cf. Satz 2.5 of part II of [15]).
(3) Let $A$ be the direct sum of two fields of two elements, that is, $A = Ia + Ib$ with

$$2A = ab = ba = a^2 - a = b^2 - b = 0.$$ 

Then $A$ satisfies condition (I). Consequently, $A$ is a restricted Boolean ring. The subgroup $K = I(a + b)$ is a subring, but $K$ is neither an ideal, nor a (ring theoretical) direct summand of $A$.

(4) Let $A$ be the direct sum of a field $B = Ib$ of order two and of a ring $C = Ic$ of order two with $c^2 = 0$. Then $A$ does not satisfy condition (I), any subring is a (ring-theoretical) direct summand of $A$, but the sub-group $I(b + c)$ is not a two-sided ideal of $A$.

(5) Let $A$ be the ring $Ia$ with $a^2 = 0$. Then $A$ is an infinite cyclic ring in which any additive subgroup is a twosided ideal with trivial multiplication, $A$ does not satisfy condition (I), and $2A$ is not a direct summand of $A$.

Remarks 6. (1) Let $C_1$ denote the class of all rings with condition (I). In the author’s paper [14] there is determined the class $C_2$ of all rings such that any subring is a (ring-theoretical) direct summand. Furthermore, Rédei [13] has determined the class $C_3$ of all rings such that any additive subgroup is a two-sided ideal. These latter rings are called full ideal rings. Now, example (3) shows that $C_1 \neq C_2$ and $C_1 \neq C_3$. Furthermore, example (4) yields $C_2 \neq C_1$ and $C_2 \neq C_3$. Finally, by example (5), we have also $C_2 \neq C_1$ and $C_2 \neq C_3$. Consequently, $C_1$, $C_2$ and $C_3$ are different classes of rings.

(2) In the proof of the implication (I) $\Rightarrow$ (II), the theorem of Forsythe and McCoy [4], according to which any regular ring without non-zero nilpotent elements is a subdirect sum of division rings, was not used.

(3) We have seen that $a = ma^2$ for any $a \in A$, where $m$ is an integer and $A$ is a ring with condition (I). This means that $a \in Ia^2$ for any $a \in A$. Let $C(a)$ denote the subgroup $Ia^2$. Condition (I) implies the $C$-regularity $a \in C(a)$ for any $a \in A$, which satisfies $C(aa) = (C(a)) \varphi$ for any (ring-theoretical) homomorphism $\varphi$ of $A$. The axiom $P_1$ of Brown and McCoy [1], p. 302, holds, but the axiom $P_2$ generally fails to be satisfied for $C(a)$. On the other hand, this $C(a)$ is a modified form of $F(a)$ of example 4 of [1], p. 308, for which axioms $P_1$ and $P_2$ are already satisfied in any ring $A$, treated as a $(F, \Omega)$-group.

(4) The upper radical $R$ (cf. Divinsky [3]), defined by the class $C_1$ of all rings with condition (I) has the property that any homomorphic image of any $R$-semisimple ring is again $R$-semisimple.

(5) It would be interesting to investigate the question, whether any finitely generated additive subgroup of any ring with condition (I) is, or is not, a direct sum of finite prime fields. (P 782)
REFERENCES


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