SOME ASPECTS OF SIMULTANEOUS RATIONAL APPROXIMATION

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This paper is the completed version of a series of lectures given during the semester. After a section on (part of) the historical developments, the attention is focused on simultaneous rational approximation with common denominator (so-called German-polynomials or type II polynomials) and the connection with recurrence relations of the Jacobi–Perron type, generalising the well-known continued fraction algorithm.

1. History

The roots lie in Number Theory: it was Ch. Hermite [103] who introduced two sets of polynomials in the study of approximation of a set of exponential functions, connected with the transcendency of e. Using the notations that are customary nowadays, the approximation problem can be formulated as follows. Let \( n \) be a natural number and consider

\[
f_0, f_1, \ldots, f_n \quad \text{n+1 formal power series in } z,
\]

\[
q_0, q_1, \ldots, q_n \quad \text{n+1 non-negative integers, } \sigma = q_0 + q_1 + \ldots + q_n.
\]

Find polynomials \( P_0, P_1, \ldots, P_n \) in \( z \), satisfying

\[
\deg P_j \leq q_j \quad (0 \leq j \leq n), \quad \sum_{j=0}^{n} P_j f_j = O(z^{\sigma+n})
\]

(so-called Latin- or type I polynomials) and an \((n+1)\)-tuple satisfying

\[
\deg P_j \leq \sigma - q_j \quad (0 \leq j \leq n), \quad P_0 f_0 - P_j f_0 = O(z^{\sigma+1}) \quad (1 \leq j \leq n)
\]

(so-called German- or type II polynomials). From a simple counting argument (number of unknowns, the coefficients of the polynomials, versus the number of equations) we infer that there always exists a non-trivial solution (i.e. not all polynomials do vanish).
For \( n = 1 \) (here type I and type II "coincide" on taking \( f_0 \equiv 1 \)) there is an enormous list of references (only several are contained in the list at the end of this paper). The name Padé table was introduced in the beginning of the twentieth century, mainly on basis of the important pioneering work done by H. Padé [156]; he also looked into the type I case for \( n \geq 2 \) in [159].

At first not so much attention was paid to the concept outside the field of Number Theory (construct rational approximants having high order of contact at the origin with certain functions, leading to an approximation of numbers by rationals too good to satisfy the famous Thue–Siegel–Roth theorem, thereby proving irrationality or even transcendency), except for an isolated generalisation into the direction of type II for a set of Stieltjes functions by A. Angelesco \([1]–[4]\). In 1934 there was a thesis by a student of Perron, J. Mall [148], a reference that surfaced a few years ago (during the continuing search by Claude Brezinski for references for his Padé bibliography), but as far as is known now, there was no follow-up from that source.

In 1934/1935 K. Mahler (finding shelter at the University of Groningen with his friend Jan Popken) wrote a long manuscript on the algebraic approximation of functions pointing out the intimate connection between the two types of polynomials. It was in 1968 that this manuscript was finally published [147] (it was the notation from this paper that led to the names German- and Latin- polynomials); the algebraic approach was continued by J. Coates [63], A. J. Goddijn [85], H. Jager [119], J. H. Loxton and A. J. van der Poorten [137], [138], A. J. van der Poorten [170], [171].

In the mean time the physicists had "rediscovered" the Padé approximant and its importance (cf. G. A. Baker jr. [12], who also wrote two books on the \( n = 1 \)-case with P. R. Graves-Morris (vol. 11 and 12) in the series "Encyclopaedia of Mathematics" edited by G.-C. Rota). It would be outside the scope of this paper to go into more detail regarding this case of the so-called "ordinary Padé table", but there are many important contributions of which only some will be mentioned here to give the reader a point of entry into the vast literature: A. I. Aptekarev [5], R. J. Arons and A. Edrei [9], A. Edrei [82] connected with [177], Peter B. Borwein [20], C. Brezinski [23], [25], M. G. de Bruin [27], A. A. Gonchar and G. Lopez L. [86], V. A. Kalyagin [125], G. Lopez L. [132]–[136] and many others, whose contributions can be found all over the literature and the fact that they are not mentioned here, has only to do with the aim of this paper to treat simultaneous approximation and does not by any means imply anything about the relevance of their work.

After 1970 the developments did run along several lines.

**Type I and type II polynomials.** A. I. Aptekarev [6], [7], A. I. Aptekarev and V. A. Kalyagin [8], G. A. Baker jr. [12], G. A. Baker jr. and D. S. Lubinsky [13], Peter B. Borwein [18], [19], M. G. de Bruin [26], [28], [29], [33], [35]–[39], D. V. Chudnovsky [51], D. V. Chudnovsky and G. V. Chudnovsky [52], G. V. Chudnovsky [59]–[61], J. Della Dora [76]–[78], J. Della Dora and
C. Di-Crescenzo [79], [80], E. M. Nikishin [151], V. N. Sorokin [186], H. Stahl [187]–[189].

**Vector-valued interpolants.** P. R. Graves-Morris [89], [90], P. R. Graves-Morris and C. D. Jenkins [92], [93], P. R. Graves-Morris and E. B. Saff [94], P. R. Graves-Morris and J. M. Wilkins [95], J. van Iseghem [107], [108], [110], [112], D. E. Roberts and P. R. Graves-Morris [174].


**Algebraic Hermite–Padé approximants.** G. A. Baker jr. and D. S. Lubinsky [13], Peter B. Borwein [18], [19], R. E. Shafer [180], H. Stahl [187]–[189].


The last two types mentioned construct polynomials of type I for the set of functions $1, f, f^2, \ldots, f^n$ resp. $1, f, f', \ldots, f^{(n)}$. In the order condition (0) on the previous page the functions are replaced by $y^j$ resp. $y^{(j)}$ and the resulting algebraic resp. differential equation (put the order term equal to zero) is solved for $y$, leading to an approximation for $f$.

The role played in Number Theory can be viewed into by consulting papers by F. Beukers [15], D. V. Chudnovsky and G. V. Chudnovsky [54]–[56], [58], G. V. Chudnovsky [62], K. Mahler [144]–[146], V. N. Sorokin [184], [185] (don’t forget the famous irrationality result on $\zeta(3)$ by Apéry) and some sidelines into the theory of differential equations by looking at D. V. Chudnovsky and G. V. Chudnovsky [53], [57].

Parallel to this development there have been several generalisations to the multivariate case cf. C. Brezinski [24], Cl. Chaffy [45], J. S. R. Chisholm [46], [47], J. S. R. Chisholm and P. R. Graves-Morris [48], J. S. R. Chisholm and R. Hugh Jones [49], J. S. R. Chisholm and J. McEwan [50], P. R. Graves-Morris [88], P. R. Graves-Morris, R. Hugh Jones and G. J. Makinson [91], R. Hugh Jones and G. J. Makinson [105] (and many other publications from the Canterbury group), A. M. Cuyt [68]–[73], A. M. Cuyt and B. M. Verdonk [74], A. M. Cuyt, H. Werner and L. Wuytack [75], G. John and C. H. Lutterodt [120], J. Karlsson and H. Wallin [126], D. Levin [130], C. H. Lutterodt [139, 140], P. Sablonniere [176], H. Werner [194], [195]. We will not look into this subject here.

Returning to the one variable case, we recollect that an important tool in the study of convergence is the study of (solutions of) recurrence relations: in the ordinary Padé table there is a fruitful connection between sequences of approximants and continued fractions! It already started with G. Frobenius [84] and there have been important results as early as for instance R. de
Montessus de Ballore [149], R. Pringsheim [172], E. B. van Vleck [191]. There have appeared several books on continued fractions cf. A. Ya. Khinchin [106] (of course also Hovanskii), W. B. Jones and W. J. Thron [121], O. Perron [167], H. S. Wall [193] (it is good to realize that usually the inverted Padé denominators — for a polynomial of degree $k$ look at $z^k P(z^{-1})$ — are nothing else but polynomials orthogonal with respect to a (sometimes indefinite) innerproduct on the space of all polynomials).

Looking at simultaneous approximants, it is obvious to try to find a generalisation of the continued fraction concept (already H. Padé [158] found all “regular algorithms” in a Padé table of type I for $n = 2$; in [159] he gave several regular algorithms in the case of general $n$). Here we have to turn to C. G. J. Jacobi [117] (the length of the recurrence relation is increased by 1 compared to the ordinary continued fraction) and O. Perron [166] for the general case. We also have to mention P. Bachmann [10], Ch. Hermite [104], S. Pincherle [168].

After the introduction of what is nowadays called the Jacobi–Perron algorithm, the interest went into several directions.

One of the main areas of interest was in Number Theory (recovering linear or algebraic dependence, transcendency, calculation of units in number fields, calculation of best (simultaneous) rational approximants, etc.) cf. A. J. Brentjes[21], [22], V. Brun [42], [43], H. R. P. Fergusson and R. W. Forcade [83], R. Güting [96]–[98], W. Jurkat, W. Kratz and A. Peyerimhoff [123], N. Pipping [169], E. S. Selmer [179], G. Szekeres [190], G. F. Voronoï [192].

The Jacobi–Perron algorithm was studied (and extended) by L. Bernstein [14], M. G. de Bruin [26], [30]–[32], [34], M. G. de Bruin and L. Jacobsen [40], [41], P. van der Cruyssen [64]–[67], E. Dubois [81], L. Jacobsen [118], P. Levrie [131], H. Padé [157]–[159], R. Paysant Le Roux and E. Dubois [165], H. Rütishauser [175], F. Schweiger [178].

Other generalisations (vector- or matrix valued, more variable-case sometimes) were treated by M. Hallin [99]–[102], J. van Iseghem [109], [111], Alphonse Magnus [141]–[143], J. A. Murphy and M. R. O’Donohue [150], V. I. Parusnikov [160]–[164].

Finally, special attention must be paid to the concept of a branched continued fraction, almost automatically leading to multivariate approximation: D. I. Bodnar [16], P. I. Bodnarchuk [17], Kh. I. Kuchminskaya [127], [128], Kh. I. Kuchminskaya and W. Siemaszko [129], V. Ya. Skorobogatko [181] — a very valuable source and important entry point for the vast amount of publications on the subject, of which only very few have been translated up to now — [182].

As the reader can see, it is necessary to restrict ourselves something in the choice of material for the sequel (l’embarras du choix): from now on we will look into the type II polynomials for general $n$ (the Padé-$n$-table) and one of their “regular algorithms” (the $C$-fraction) only. For the proofs the reader is referred to the references (a.o. [26]–[41]).
2. The Padé-\(n\)-table

Consider \(n\) (\(n \geq 1\)) formal power series in \(z\) with complex coefficients

\[
f_j(z) = \sum_{k=0}^{\infty} c_k^{(j)} z^k, \quad c_k^{(j)} \neq 0 \quad (1 \leq j \leq n),
\]

(the condition that the constant terms do not vanish has been added for sake of simplicity only, a simple change in the definitions in the sequel will make it possible to treat those cases where one (or more) functions vanish at the origin.)

Define for the non-negative numbers \(q_0, q_1, \ldots, q_n\), with \(\sigma = q_0 + q_1 + \ldots + q_n\), for \(j = 1, 2, \ldots, n\) the “building blocks” \(D^{(j)} = D^{(j)}(q_0, q_1, \ldots, q_n)\) by

\[
(2a) \quad D^{(j)} = \begin{bmatrix}
  c_{\sigma-q_0}^{(j)} & c_{\sigma-q_1}^{(j)} & c_{\sigma-q_2}^{(j)} & \ldots & c_{\sigma-q_n}^{(j)} \\
  c_{\sigma-q_1}^{(j)} & c_{\sigma-q_2}^{(j)} & c_{\sigma-q_3}^{(j)} & \ldots & c_{\sigma-q_n}^{(j)} \\
  c_{\sigma-q_2}^{(j)} & c_{\sigma-q_3}^{(j)} & c_{\sigma-q_4}^{(j)} & \ldots & c_{\sigma-q_n}^{(j)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{\sigma-q_n}^{(j)} & c_{\sigma-q_{n-1}}^{(j)} & c_{\sigma-q_{n-2}}^{(j)} & \ldots & c_{\sigma-q_0}^{(j)}
\end{bmatrix},
\]

\((1 \leq j \leq n)\),

(for \(q_j = 0\) the block is empty) and the “augmented” blocks \(\hat{D}^{(j)} = \hat{D}^{(j)}(q_0, q_1, \ldots, q_n)\) by

\[
(2b) \quad \hat{D}^{(j)} = \begin{bmatrix}
  c_{\sigma-q_0}^{(j)} & c_{\sigma-q_1}^{(j)} & c_{\sigma-q_2}^{(j)} & \ldots & c_{\sigma-q_n}^{(j)} \\
  c_{\sigma-q_1}^{(j)} & c_{\sigma-q_2}^{(j)} & c_{\sigma-q_3}^{(j)} & \ldots & c_{\sigma-q_n}^{(j)} \\
  c_{\sigma-q_2}^{(j)} & c_{\sigma-q_3}^{(j)} & c_{\sigma-q_4}^{(j)} & \ldots & c_{\sigma-q_n}^{(j)} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{\sigma-q_n}^{(j)} & c_{\sigma-q_{n-1}}^{(j)} & c_{\sigma-q_{n-2}}^{(j)} & \ldots & c_{\sigma-q_0}^{(j)} \\
  c_{\sigma-q_1}^{(j)} & c_{\sigma-q_0}^{(j)} & c_{\sigma-q_{-1}}^{(j)} & \ldots & c_{\sigma-q_{n+1}}^{(j)}
\end{bmatrix},
\]

Furthermore we introduce the \((\sigma - q_0) \times (\sigma - q_0)\) determinant \(D = D(q_0, q_1, \ldots, q_n)\) by

\[
(3a) \quad D = \det \begin{bmatrix}
  D^{(1)} \\
  \vdots \\
  D^{(n)}
\end{bmatrix},
\]

and the \((\sigma - q_0 + n) \times (\sigma - q_0 + 1)\) matrix \(\hat{D}\) by

\[
(3b) \quad \hat{D} = \begin{bmatrix}
  \hat{D}^{(1)} \\
  \vdots \\
  \hat{D}^{(n)}
\end{bmatrix}.
\]

Throughout this paper it is tacitly assumed that \(c_k^{(j)} = 0\) for \(k < 0\) and that empty building blocks are omitted.

Consider now the type II problem of finding polynomials

\[
P_j(z) = P_j(q_0, q_1, \ldots, q_n; z) \quad (j = 0, 1, \ldots, n),
\]
satisfying

\[(4a) \quad \deg P_j(z) \leq \sigma - q_j \quad (j = 0, 1, \ldots, n),\]

\[(4b) \quad P_0(z) f_j(z) - P_j(z) = O(z^{\sigma+1}) \quad (j = 0, 1, \ldots, n).\]

Then we have the following theorem:

**Theorem 1.** If \( D(q_0, q_1, \ldots, q_n) \neq 0 \), then the solution of (4a, b), which is unique after the normalization \( P_0(q_0, q_1, \ldots, q_n; 0) = 1 \), is given by

\[
P_0(z) = \frac{1}{D} \det \begin{bmatrix} 1 & z & z^2 & \cdots & z^{\sigma - q_0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\sigma - q_j + 1}^{(f)} & c_{\sigma - q_j}^{(f)} & c_{\sigma - q_j - 1}^{(f)} & \cdots & c_{\sigma - q_j + 1}^{(f)} \\ c_{\sigma - q_j + 2}^{(f)} & c_{\sigma - q_j + 1}^{(f)} & c_{\sigma - q_j}^{(f)} & \cdots & c_{\sigma - q_j}^{(f)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\sigma}^{(f)} & c_{\sigma - 1}^{(f)} & c_{\sigma - 2}^{(f)} & \cdots & c_{\sigma}^{(f)} \end{bmatrix}.
\]

\[
P_i(z) = \frac{1}{D} \sum_{k=1}^{\sigma - q_i} \det \begin{bmatrix} c_k^{(f)} & c_{k-1}^{(f)} & c_{k-2}^{(f)} & \cdots & c_{\sigma - q_0}^{(f)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\sigma - q_j + 1}^{(f)} & c_{\sigma - q_j}^{(f)} & c_{\sigma - q_j - 1}^{(f)} & \cdots & c_{\sigma - q_j + 1}^{(f)} \\ c_{\sigma - q_j + 2}^{(f)} & c_{\sigma - q_j + 1}^{(f)} & c_{\sigma - q_j}^{(f)} & \cdots & c_{\sigma - q_j}^{(f)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\sigma}^{(f)} & c_{\sigma - 1}^{(f)} & c_{\sigma - 2}^{(f)} & \cdots & c_{\sigma}^{(f)} \end{bmatrix} z^k \quad (1 \leq i \leq n),
\]

\[
P_0(z) f_i(z) - P_i(z) = \frac{1}{D} \sum_{k=1}^{\infty} \det \begin{bmatrix} c_{\sigma + k}^{(f)} & c_{\sigma + k - 1}^{(f)} & c_{\sigma + k - 2}^{(f)} & \cdots & c_{\sigma + k}^{(f)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\sigma - q_j + 1}^{(f)} & c_{\sigma - q_j}^{(f)} & c_{\sigma - q_j - 1}^{(f)} & \cdots & c_{\sigma - q_j + 1}^{(f)} \\ c_{\sigma - q_j + 2}^{(f)} & c_{\sigma - q_j + 1}^{(f)} & c_{\sigma - q_j}^{(f)} & \cdots & c_{\sigma - q_j}^{(f)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{\sigma}^{(f)} & c_{\sigma - 1}^{(f)} & c_{\sigma - 2}^{(f)} & \cdots & c_{\sigma}^{(f)} \end{bmatrix} z^{\sigma + k} \quad (1 \leq i \leq n).
\]

From now on we will assume that we are always in the situation of having a unique solution after the normalisation used before.

**Definition 1.** The \( n \)-tuple \((f_1, f_2, \ldots, f_n)\) will be called **regular** if

\[(8a) \quad D(q_0, q_1, \ldots, q_n) \neq 0 \quad \text{for all} \quad (q_0, q_1, \ldots, q_n),\]
and semi-regular if

\[(8b) \quad D(\varrho_0, \varrho_1, \ldots, \varrho_n) \neq 0 \quad \text{for all } (\varrho_0, \varrho_1, \ldots, \varrho_n) \text{ with } \varrho_0 \geq \varrho_j - 1 \quad (1 \leq j \leq n).\]

Placing at each point with non-negative integer coordinates in \((n+1)\)-space the unique solution \((5, 6)\) of problem \((4a, b)\), we arrive at the configuration that will be referred to as the Padé-\(n\)-table; for a semi-regular \(n\)-tuple of functions only the "upper half" of the table is defined. Introduce the concept of normality in the same way as was done for the ordinary Padé table (this is actually the case \(n = 1\) in the preceding formulæ) by

**DEFINITION 2.** The point \((\varrho_0, \varrho_1, \ldots, \varrho_n)\) in the Padé-\(n\)-table for the \(n\)-tuple \((f_1, f_2, \ldots, f_n)\) is called normal if the solution \((5, 6)\) belonging to the point, does not appear at any other point in the Padé-\(n\)-table.

An important result is now the following:

**THEOREM 2.** For a point \((\varrho_0, \varrho_1, \ldots, \varrho_n)\) in the Padé-\(n\)-table for a (semi-) regular \(n\)-tuple of formal power series the following properties are equivalent:

(a) \((\varrho_0, \varrho_1, \ldots, \varrho_n)\) is normal.

(b) \(\deg P_j(z) = \sigma - \varrho_j (0 \leq j \leq n), \quad P_0(z)f_j(z) - P_j(z) = O(z^{\sigma+2})\) for at least one \(j \in \{1, 2, \ldots, n\}\).

(c) The determinants \(D(\varrho_0, \varrho_1, \ldots, \varrho_n), D(\varrho_0 + 1, \varrho_1, \ldots, \varrho_n), D(\varrho_0, \varrho_1, \ldots, \varrho_{j-1}, \varrho_{j+1}, \varrho_{j+1}, \ldots, \varrho_n)\) \((1 \leq j \leq n)\) are different from zero and rank \(D = \sigma - \varrho_0 + 1\).

Very few situations of completely regular or normal Padé-\(n\)-tables are known, most of them appear in the following list:

(a) the exponential function system \((e^{\lambda z}, 1 \leq j \leq n)\) with \(\lambda_j \neq 0, \lambda_j \neq \lambda_i\); normal, explicit formulæ known (Ch. Hermite, H. Jager).

(b) the binomial function system \((\lambda_1, \ldots, \lambda_n, 1 \leq j \leq n)\) with \(\lambda_j \neq \lambda_i\); normal, explicit formulæ known (H. Jager).

(c) the logarithmic function system \((\log(1-z), 1 \leq j \leq n)\) with \(\varrho_0 \leq \varrho_1 \leq \ldots \leq \varrho_n\); normal (H. Jager; (b) and (c) together in one normality theorem by A. Baker).

(d) Angelesco-systems; explicit formulæ (A. Angelesco, E. M. Nikishin, V. N. Sorokin).

(e) the hypergeometric function systems \(\binom{a_j}{c} F_1(1; a_j + b; z), 1 \leq j \leq n\) with \(a_j, a_j + b \notin \mathbb{Z} \setminus \mathbb{N}, a_i - a_j \notin \mathbb{Z} ; \binom{a_1}{c_1} F_1(1; c_j; z), 1 \leq j \leq n\) with \(c_j \notin \mathbb{Z} \setminus \mathbb{N}, c_i - c_j \notin \mathbb{Z} ; \binom{a_1}{c_1} F_0(a_j, 1; z), 1 \leq j \leq n\) with \(a_j \notin \mathbb{Z} \setminus \mathbb{N}, a_i - a_j \notin \mathbb{Z} ;\) semi-regular, normal for \(\varrho_0 \geq \varrho_1\), for the second and third system explicit formulæ (M. G. de Bruin).

(f) the hypergeometric function system \(\binom{a}{c} F_1(1; c; \lambda_j z), 1 \leq j \leq n\) with \(\lambda_j \neq 0, \lambda_j \neq \lambda_i\); semi-regular, explicit formulæ known (A. I. Aptekarev).

(g) the \(q\)-hypergeometric function systems \(\binom{A}{c} F_1((1, \ldots, 1; (C, \gamma_j)); z), 1 \leq j \leq n\) with \(A \neq q^{\gamma_j + \delta \lambda_j}, C \neq q^{\gamma_j + \delta \lambda_j}, c_i \neq a_i \neq \mathbb{Z} ; q^k \neq Aq^k \quad \text{for} \quad k \geq 0, \quad q^{\gamma_j - \gamma_k + \delta \lambda_j} \neq 1 \quad (i \neq j)\) for \(k \in \mathbb{Z} ; \binom{A}{c} F_1((1, 1; (C, \gamma_j)); z), 1 \leq j \leq n\) with \(C \neq q^{\gamma_j + \delta \lambda_j} \quad \text{for} \quad k \geq 0, \quad q^{\gamma_j - \gamma_k + \delta \lambda_j} \neq 1 \quad (i \neq j)\).
\[ q^{u_n-n+k} \neq 1 \quad (i \neq j) \quad \text{for} \quad k \in \mathcal{Z}, \quad (z^0, (A, \alpha), (1, 1); z), \quad 1 \leq j \leq n, \quad \text{with} \quad A \neq q^{\alpha_j + k} \quad \text{for} \quad k \geq 0, \quad q^{\alpha_j - \alpha_j + k} \neq 1 \quad (i \neq j) \quad \text{for} \quad k \in \mathcal{Z}; \quad \text{in all} \quad 3 \quad \text{cases} \quad q \in \mathcal{C} \setminus \{0, 1\}, \quad q^k \neq 1 \quad (k \geq 1), \quad \text{semi-regular, normal for} \quad q_0 \geq q_j, \quad \text{for the second and third system explicit formulae (M. G. de Bruin)}.

In the last example the \( q \)-hypergeometric functions are introduced according to the following notation:

\[
{_{s} \Phi_{3}}((A_1, \alpha_1), \ldots, (A_r, \alpha_r); (C_1, \gamma_1), \ldots, (C_s, \gamma_s); z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{r} (A_i, \alpha_i; q)_k}{\prod_{j=1}^{s} (C_j, \gamma_j; q)_k} (1, 1; q)_k z^k,
\]

with the generalization of the ascending factorial (Pochhammer's symbol) given by

\[(A, \alpha; q)_0 = 1, \quad (A, \alpha; q)_n = (A - q^n)(A - q^{n+1}) \cdots (A - q^{n+n-1}) \quad (n \geq 1).
\]

One might wonder how the block structure of the ordinary Padé table is translated for the type of multidimensional table that has just been defined. A simple example shows the problem we have to face.

Suppose that the point \((q_0, q_1, \ldots, q_n)\) is normal and has as solution to problem (4a, b) a set of polynomials with actual degrees \((p_0, p_1, \ldots, p_n)\), then automatically \(\sigma - q_j = p_j\) for \(1 \leq j \leq n\). Adding all the equations and inserting the definition of \(\sigma\), we find the condition

\[ p_0 + p_1 + \ldots + p_n \equiv 0 \quad (\text{mod} \quad n). \]

The difference between the case \(n = 1\) and \(n \geq 2\) is obvious: for the ordinary Padé table this is no restriction at all! When \(n \geq 2\), however, we have to put \(p_0 + p_1 + \ldots + p_n = kn\), to satisfy the condition and we find \(q_j = k - p_j\), \(1 \leq j \leq n\). Here we see another condition on the \(p\)'s:

\[ n \max_{0 \leq j \leq n} p_j \leq p_0 + p_1 + \ldots + p_n. \]

(note that this problem of having conditions on the \(p\)'s does not also arise in the case of the type I polynomials.)

Looking better into the subject, it is a matter of simple inequalities to prove the following theorem

**Theorem 3.** Let \(\{P_j(z)/P_0(z), \quad 1 \leq j \leq n\}\) appear at \((q_0, q_1, \ldots, q_n)\) in the Padé-\(n\)-table for an \(n\)-tuple of formal power series such that (4a, b) has a unique solution up to a multiplicative constant. Write this solution in its simplest terms and define

\[ \text{GCD}(P_0, P_1, \ldots, P_n) = 1; \quad P_0(0) = 1, \quad P_j(0) = c_j^0 \quad (1 \leq j \leq n), \]

\[ \text{deg} P_j = p_j \quad (1 \leq j \leq n), \quad P_0 f_j - P_j = d_j z^{k_j} + \ldots, \quad d_j \neq 0, \quad k_j < \infty \quad (1 \leq j \leq n), \]

\[ r_j \quad \text{follows from} \quad nk_j = p_0 + p_1 + \ldots + p_n + r_j, \quad r_j \geq 0 \quad (1 \leq j \leq n). \]
Finally define

\[ r = \min_{1 \leq j \leq n} r_j. \]

Then we have

(a) If \( p_0 + p_1 + \ldots + p_n \equiv 0 \) (mod n) and \( q_j = (p_0 + p_1 + \ldots + p_n)/n - p_j \), \( 1 \leq j \leq n \), then

\[ (q_0, q_1, \ldots, q_n) \text{ is normal } \iff r = 0. \]

(b) If \( r \geq 1 \), the set to which the n-tuple belongs, consists of at least two points. They are just the points having all coordinates non-negative, taken from the following sets.

A. For \( p_0 + p_1 + \ldots + p_n = mn \) with \( m \geq 0 \) fixed:

\[ (\tilde{q}_0 + v_0, \tilde{q}_1 + v_1, \ldots, \tilde{q}_n + v_n) \text{ with } \tilde{q}_j = m - p_j \quad (0 \leq j \leq n), \ r = nv \ (v \geq 1) \]

and (1) \( v_0 + v_1 + \ldots + v_n = w \), (2) \( \max v_j \leq \min (w, v) \), (3) \( \min v_j \geq w - nv \) where \( 0 \leq w \leq (n+1)v \).

B. For \( p_0 + p_1 + \ldots + p_n = mn + n-k \) with \( 1 \leq k \leq n-1 \), \( m \geq 0 \) both fixed:

\[ (\tilde{q}_0 + v_0, \tilde{q}_1 + v_1, \ldots, \tilde{q}_n + v_n) \text{ with } \tilde{q}_j = m - p_j \quad (0 \leq j \leq n), \ r = nv + k (v \geq 0) \]

and (1) \( v_0 + v_1 + \ldots + v_n = w + n - k + 1 \), (2) \( \max v_j \leq \min (w, v) + 1 \), (3) \( \min v_j \geq w - nv - k + 1 \) where \( 0 \leq w \leq (n+1)v + k \).

Remark. The conditions follow from the fact that there should exist a non-negative integer \( x \), such that \( x + p_j \leq \sigma - q_j \) \( (0 \leq j \leq n) \), \( x + k_j \geq \sigma + 1 \) \( (1 \leq j \leq n) \).

Adding the last set of inequalities, inserting \( n \sigma = (\sigma - q_0) + (\sigma - q_1) + \ldots + (\sigma - q_n) \), and inserting the first set of inequalities, we find \( r_j \geq 0 \) and with \( r = \min r_j \) it turns out that the coordinates of the points where the rational functions appear should satisfy \( p_j + x \leq \sigma - q_j \leq p_j + r \) \( (0 \leq j \leq n) \).

For \( n = 2 \) (a three dimensional table) we easily deduce the following pictures:
Now we turn our attention towards walks in the table. Restricting ourselves to the case \( n = 2 \) for sake of simplicity, an algorithm connected with the path \((q_0(k), q_1(k), q_2(k))\) will be called regular (cf. [158]) if we have the following situation

(a) the points \((q_0(k), q_1(k), q_2(k))\) admit a unique solution to problem (4a, b),

(b) \(\sigma(k) = q_0(k) + q_1(k) + q_2(k)\) is strictly increasing in \(k\),

(c) the determinants

\[
\det \begin{bmatrix}
P_0(k; z) & P_0(k+1; z) & P_0(k+2; z) \\
P_1(k; z) & P_1(k+1; z) & P_1(k+2; z) \\
P_2(k; z) & P_2(k+1; z) & P_2(k+2; z)
\end{bmatrix}
\]

all are monomials in \(z\),

(d) the three sequences of polynomials satisfy the same recurrence relation

\[P_j(k+3; z) = a_k(z) P_j(k+2; z) + b_k(z) P_j(k+1; z) + c_k(z) P_j(k; z)\]

\((j = 0, 1, 2; k \geq 1)\),

where the coefficients are polynomials in \(z\) of fixed degrees and orders.

Using the same method as H. Padé did, it is possible to find all regular algorithms in the Padé-2-table. It turns out that there are basically only three regular algorithms in the table (although of course relaxation of the conditions leads to other, quite interesting algorithms: for instance when walking along a path of constant \(\sigma\), i.e. an anti-diagonal, cf. [95]) which are given below

A. coefficients \(1, \beta_k z, \gamma_k z^2\) with \(\gamma_k \neq 0\); a generalised stepline, the coordinates of the points show a relative increase of \((1, 0, 0), (0, 1, 0), (0, 0, 1)\) ad inf. (a C-2-fraction).

B. coefficients \(1 + \alpha_k z, \beta_k z, \gamma_k z^2\) with \(\gamma_k \neq 0\); an ordinary stepline in a plane perpendicular to one of the axes, the coordinates of the points show a relative increase of \((1, 0, 0), (0, 1, 0)\) or \((1, 0, 0), (0, 0, 1)\) or \((0, 1, 0)\), \((0, 0, 1)\) ad inf.

C. coefficients \(1 + \alpha_k z, \beta_k z + \gamma_k z^2, \delta_k z^2\) with \(\delta_k \neq 0\); a walk parallel to one of the axes, the coordinates of the points show a relative increase of \((1, 0, 0)\) or \((0, 1, 0)\) or \((0, 0, 1)\) ad inf.

The method of deriving the result quoted above, shows that there is no hope to achieve anything when solving the problem for general \(n\) in that way (the first step is to find all admissible triplets of points (a) that satisfy (b, c), combine these amongst themselves to find all quadruplets of points such that the first three and the last three points both satisfy (b, c) — this already leads to 261 quadruplets, giving 87 basic situations if we allow for cyclic permutations of the coordinates — and finally the pattern for the coefficients of the recurrence relation has to be calculated; this gives 6 different patterns, of which only 3 satisfy the requirements for regularity given above).
In the Padé-$n$-table there exist of course variants on the algorithms B and C (this time there are more choices, but the reader has to bear in mind, that keeping one or more of the coordinates fixed — barring the first coordinate $\varphi_0$ — means nothing else than *not using the function(s) having the same index as the coordinate(s): we are in the situation of a table for $m < n$ functions!*)

Now the restriction will be made to one special type of algorithm: the generalisation of the ordinary stepline (type A given before). It is possible to prove the following theorem.

**Theorem 4.** Let $k$ be a non-negative integer and let the stepline

$$(k, 0, \ldots, 0), (k + 1, 0, \ldots, 0), (k + 1, 1, 0, \ldots, 0), \ldots, (k + 1, 1, \ldots, 1), (k + 2, 1, \ldots, 1), (k + 2, 2, 1, \ldots, 1), \ldots$$

in the Padé-$n$-table for the functions

$$f_j(z) = \sum_{k=0}^{\infty} c_k^{(j)} z^k \quad (1 \leq j \leq n)$$

be normal. Then the polynomials $P_0(m; z), P_1(m; z), \ldots, P_n(m; z)$ (the points are numbered consecutively $0, 1, 2, \ldots, n, n+1, n+2, \ldots$) all satisfy the same recurrence relation

$$(9a) \quad X_m = X_{m-1} + a_m^{(n)} X_{m-2} + a_m^{(n-1)} X_{m-3} + \ldots + a_m^{(1)} X_{m-n-1} \quad (m \geq 1),$$

with for each sequence a different set of initial values

$$(9b) \quad P_j(-m; z) = \delta_{m+j,n+1} \quad (0 \leq j \leq n; 1 \leq m \leq n),$$

$$P_0(0; z) = 1, \quad P_j(0; z) = \sum_{i=0}^{k} c_i^{(j)} z^i \quad (1 \leq j \leq n).$$

The coefficients in the recurrence relation have the form

$$(9c) \quad a_m^{(j)} = \begin{cases} a_{m,j} z^{k+m} & (1 \leq m \leq j \leq n), \\ a_{m,j} z^{\min(n+1-j,m)} & (1 \leq j \leq m \leq n), \\ a_{m,j} z^{n+1-j} & (j \geq n+1), \end{cases}$$

where the constants in the monomials satisfy $a_{j,m} \in \mathbb{C}$ and $a_{1,m} \neq 0 \ (m \geq 1)$.

Here we see clearly the connection with recurrence relations of length greater than or equal to three; this subject will be treated in the next section. As a final remark — a matter which will not be elaborated upon — it must be mentioned that the concept of Padé approximation can be introduced through so-called Padé-type-approximants, using a straightforward generalisation of the theory of C. Brezinski [23] (cf. [37]).
3. Generalised Continued Fractions

The type of generalisation to be considered here is inspired by the Jacobi–Perron algorithm, which actually stems from a Euclidean algorithm for the simultaneous approximation of \( n \) real numbers by \( n \) rational numbers with a common denominator.

There are many ways to introduce the concept that plays the main role in this section, due to space limitations only one method will be given; for other methods (consecutive linear fractional transformations in \( n \) variables etc.) the reader can consult for instance [26], [30].

Let \( n+1 \) sequences of complex numbers be given

\[
b_m, a_m^{(n)}, a_m^{(n-1)}, \ldots, a_m^{(1)} \quad (m \geq 1), \quad b_m \neq 0 \quad (m \geq 1),
\]

along with \( n \) starting values

\[
b_0^{(n)}, b_0^{(n-1)}, \ldots, b_0^{(1)}.
\]

The set of data will usually be given by the notation

\[
\begin{bmatrix}
a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & \cdots \\
b_0^{(1)} & a_1^{(2)} & a_2^{(2)} & a_3^{(2)} & \cdots \\
b_0^{(2)} & a_1^{(3)} & a_2^{(3)} & a_3^{(3)} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
b_0^{(n-1)} & a_1^{(n)} & a_2^{(n)} & a_3^{(n)} & \cdots \\
b_0^{(n)} & b_1 & b_2 & b_3 & \cdots & b_m & \cdots
\end{bmatrix}
\]

(10)

**Definition 3.** Given a set of data (10), an \( n \)-fraction is defined by its sequence of \( n \)-tuples of approximants \( (A_m^{(j)}, A_m^{(0)}), \ 1 \leq j \leq n, \ m \geq 0 \). All sequences \( (A_m^{(j)}, m \geq 0), \ 0 \leq j \leq n, \) satisfy the same recurrence relation

\[
X_m = b_m X_{m-1} + a_m^{(n)} X_{m-2} + a_m^{(n-1)} X_{m-3} + \ldots + a_m^{(1)} X_{m-n} \quad (m \geq 1),
\]

with for each sequence a different set of initial values

\[
A_m^{(j)} = \delta_{m+j,n+1} \quad (0 \leq j \leq n; \ 1 \leq m \leq n),
\]

\[
A_0^{(1)} = 1, \quad A_0^{(j)} = b_0^{(j)} \quad (1 \leq j \leq n).
\]

We then have the formalism to calculate the numerators and denominators (for the moment only formally, existence and convergence will be treated lateron) using matrix-multiplication. Introducing the following matrices

\[
\mathbf{A}_0 =
\begin{bmatrix}
1 & 0 & 0 & 0 \\
b_0^{(1)} & 1 & 0 & 0 \\
b_0^{(2)} & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
b_0^{(n)} & 0 & 0 & 1
\end{bmatrix}
\]
\[ B_m = \begin{bmatrix} b_m & 0 & 0 \\ d_m^{(m)} & 0 & 1 \\ d_m^{(2)} & 0 & 0 \\ d_m^{(1)} & 0 & 0 \end{bmatrix}, \quad A_m = A_{m-1} B_m \quad (m \geq 1), \]

It is a simple matter of mathematical induction to prove

\[ A_m = \begin{bmatrix} A_m^{(0)} & \ldots & A_m^{(0)} \\ \vdots & \ddots & \vdots \\ A_m^{(n)} & \ldots & A_m^{(n)} \end{bmatrix}. \]

The concept of a terminating \( n \)-fraction is now defined easily: there is just a finite set of data, terminating with a certain index, say \( k \). The sequences of denominators and numerators — defining the approximants — then are also terminated after index \( k \); the value of a (non) terminating \( n \)-fraction is then nothing else but the \( n \)-tuple of values \( A_k^{(j)}/A_k^{(0)} \), \( 1 \leq j \leq n \) (or — in the non-terminating case — the limit for \( k \to \infty \) of these expressions). Fundamental for existence/convergence is the following theorem.

**Theorem 5.** Consider a non-terminating \( n \)-fraction (10) with finite limits. If \( d_1^{(1)}, d_2^{(1)}, \ldots, d_k^{(1)} \neq 0 \), then any two of the following formulae imply the third:

\[ \begin{bmatrix} \xi_0^{(1)} \\ \vdots \\ \xi_0^{(n)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} & \ldots & a_{k-1}^{(1)} & a_k^{(1)} \\ b_0^{(1)} & a_1^{(2)} & a_{k-1}^{(2)} & \xi_k^{(1)} \\ \vdots & \vdots & \vdots & \vdots \\ b_0^{(n-1)} & a_1^{(n)} & a_{k-1}^{(n)} & \xi_k^{(n-1)} \\ b_0^{(n)} & b_1 & b_{k-1} & \xi_k^{(n)} \end{bmatrix}, \]

(11a)

\[ \begin{bmatrix} \xi_k^{(1)} \\ \vdots \\ \xi_k^{(n)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} & \ldots & a_{k-1}^{(1)} & a_k^{(1)} \\ a_2^{(2)} & a_{k-1}^{(2)} & a_m^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ a_k^{(n)} & a_{k+1}^{(n)} & a_m^{(n)} \\ b_k & b_{k+1} & \ldots & b_m \end{bmatrix}, \]

(11b)

\[ \begin{bmatrix} \xi_0^{(1)} \\ \xi_0^{(2)} \\ \vdots \\ \xi_0^{(n)} \end{bmatrix} = \begin{bmatrix} a_1^{(1)} & a_2^{(1)} & a_m^{(1)} & \ldots \\ b_0^{(1)} & a_1^{(2)} & a_2^{(2)} & a_m^{(2)} \\ \vdots & \vdots & \vdots & \vdots \\ b_0^{(n-1)} & a_1^{(n)} & a_2^{(n)} & a_m^{(n)} \\ b_0^{(n)} & b_1 & b_2 & b_m \end{bmatrix}. \]

(11c)
There is an important connection between convergence of these \( n \)-fractions and the behaviour of the solution space of the linear recurrence relation that plays the key-role in the definition of the approximant \( n \)-tuples. We have the following theorem due to P. van der Cruyssen [64].

**Theorem 6.** Consider the recurrence relation

\[
X_k = b_k X_{k-1} + a_k^{(0)} X_{k-2} + a_k^{(n-1)} X_{k-3} + \ldots + a_k^{(1)} X_{k-n-1},
\]

with \( a_k^{(1)} \neq 0 \) \( (k \geq 1) \).

The solution space is an \((n+1)\)-dimensional linear space over the complex numbers and a basis can be given by those solutions that for instance satisfy \( A_k^{(0)} = \delta_{k+j,n+1} \) \((0 \leq k \leq n, 1 \leq j \leq n+1)\). Then the fractions \( (A_k^{(j)}/A_k^{(n+1)}), 1 \leq j \leq n, k \geq 0 \), we use suffix \( n+1 \) here instead of 0, are the approximant \( n \)-tuples of the \( n \)-fraction

\[
\begin{bmatrix}
  a_1^{(1)} & a_2^{(1)} & a_k^{(1)} & \ldots \\
  0 & a_1^{(2)} & a_2^{(2)} & a_k^{(2)} & \ldots \\
  & \ldots & \ldots & \ldots & \ldots \\
  0 & a_1^{(n)} & a_2^{(n)} & a_k^{(n)} & \ldots \\
  0 & b_1 & b_2 & b_k & \ldots
\end{bmatrix}
\]

(13)

and the following statements are equivalent:

(a) There exists a dominant solution of (12) (i.e. \( X_k/D_k \to 0 \) for \( k \to \infty \) for all sequences \( (X_k) \) in an \( n \)-dimensional subspace of the solution space), for which the dominated subspace has a basis \( (X_k^{(1)}), (X_k^{(2)}), \ldots, (X_k^{(n)}) \), satisfying

\[
\begin{vmatrix}
  X_1^{(1)} & \ldots & X_n^{(1)} \\
  \ldots & \ldots & \ldots \\
  X_1^{(n)} & \ldots & X_n^{(n)}
\end{vmatrix} \neq 0.
\]

(b) The \( n \)-fraction (13) converges to finite limits

\[
\lim_{k \to \infty} \frac{A_k^{(j)}}{A_k^{(n+1)}} = \xi^{(j)} \in \mathbb{C} \quad (1 \leq j \leq n).
\]

Here we see a method to derive convergence results for an \( n \)-fraction (13) from the knowledge about the solution space of a linear recurrence relation (12) and vice versa.

The reader might wonder why the type of continued fraction given before is of interest for simultaneous approximation in the Padé sense. The answer to this has already been given in Theorem 4: approximants along a generalised stepline in a normal Padé-\( n \)-table give rise to an \( n \)-fraction! Therefore it is now about time to introduce the analytic aspects of generalised continued fractions and this will be done together with the algorithmic aspects of the correspon-
dence between $n$-tuples of formal power series and a generalisation of the
C-fraction algorithm (cf. [106], [121], [167], [193] for the ordinary C-fraction).

Consider an $n$-tuple of formal power series $f^{(1)}_k, f^{(2)}_k, \ldots, f^{(n)}_k$ with complex
coefficients (here the notation has been changed slightly from what happened
up to now: the suffix is used to number the functions in an $n$-tuple and the
index is used to number the steps in the algorithm).

Step 0. We start with $k = 0$.

(a) Take the first and second non-zero terms in $f_0^{(1)}$ — in case this power
series is a monomial or identically zero, we turn to “interruption handling” —
denote these terms by $b_{1,0} z^{r(1,0)}$ resp. $a_{1,1} z^{r(1,1)}$. Use these to define $f_1^{(n)}$ by
inverting the tail of the function

$$f_0^{(1)}(z) = b_{1,0} z^{r(1,0)} + \frac{a_{1,1} z^{r(1,1)}}{f_1^{(n)}(z)}, \quad 0 \leq r(1, 0) < r(1, 1), f_1^{(n)}(z) = 1 + O(z).$$

(b) Take the first non-zero term of $f_0^{(j)}$ and force the tail to have the same
denominator as in (a), thereby defining $f_1^{(j-1)}$, $2 \leq j \leq n$,

$$f_0^{(j)}(z) = b_{j,0} z^{r(j,0)} + \frac{f_1^{(j-1)}(z)}{f_1^{(n)}(z)},$$

$$0 \leq r(j, 0), f_1^{(j-1)}(z) = O(z^p) \text{ with } p > r(j-1, 0).$$

We now have an $n$-tuple of functions with index 1 and $f_1^{(n)}(z) = 1 + O(z)$.

Step $k$. We start from an $n$-tuple $f^{(1)}_k, f^{(2)}_k, \ldots, f^{(n)}_k$.

A. $f^{(1)}_k(z)$ is not a monomial nor identically zero.

(a) As in step 0, we use the function with suffix 1, index $k$, to define the the
function with suffix $n$, index $k + 1$

$$f^{(1)}_k(z) = a_{2,k} z^{r(2,k)} + \frac{a_{1,k+1} z^{r(1,k+1)}}{f_{k+1}^{(n)}(z)},$$

$$1 \leq r(2, k) < r(1, k + 1), f_{k+1}^{(n)}(z) = 1 + O(z).$$

(b) As in step 0, the functions with suffix $j$, index $k$, are used to define the functions with suffix $j - 1$, index $k + 1$.

$$f^{(j)}_k(z) = a(j + 1, k) z^{r(j + 1,k)} + \frac{f^{(j-1)}_{k+1}(z)}{f_{k+1}^{(n)}(z)},$$

$$f^{(j-1)}_{k+1}(z) = O(z^p) \text{ with } p > r(j + 1, k) (2 \leq j \leq n - 1),$$

$$f^{(n)}_{k+1}(z) = 1 + \frac{f^{(n-1)}_{k+1}(z)}{f_{k+1}^{(n)}(z)}, \quad f^{(n-1)}_{k+1}(z) = O(z^p) \text{ with } p > r(n, k).$$

B. $f^{(1)}_k(z)$ is a monomial or identically zero.

The first function cannot be used to define the denominator as in the
previous cases. We assume that $f^{(1)}_k, f^{(2)}_k, \ldots, f^{(n)}_k$ all are monomials or
identically zero. But $j^{(s+1)}_k$ is the first power series having at least two terms. This is called an interruption of order $s$ at index $k$. There are now two possibilities.

B1. It happens that $s = n$: all power series have at most one non-zero term. Put $a_{j+1,k} z^{r(j+1,k)} = f^{(j)}_k(z)$ (1 $\leq j \leq n-1$), $b_k = 1$ — the term $a_{1,k} z^{r(1,k)}$ has already been found in the previous step — and the algorithm terminates. Calculating backwards, we find that the original series must be the Taylor series of an $n$-tuple of rational functions for which $z = 0$ is a regular point in the complex plane.

B2. Now 1 $\leq s < n-1$. Put $a_{j+1,k} z^{r(j+1,k)} = f^{(j)}_k(z)$ (1 $\leq j \leq s$), $a_{j,m} z^{r(j,m)} = 0$ (1 $\leq j \leq s$; $m \geq k+1$), and define $f^{(s+1)}_k$ using two non-zero terms from $f^{(s+1)}_k$.

Now repeat case A for the functions that are left over, i.e. for $f^{(s+1)}_k$, $f^{(s+2)}_k$, ..., $f^{(n)}_k$.

The algorithm now goes on until another interruption occurs; either it terminates or it does not, the result is called a C-n-fraction. A compact notation for the result of the algorithm is given below:

$$
\begin{bmatrix}
  f^{(1)}_0(z) \\
  \vdots \\
  f^{(n)}_0(z)
\end{bmatrix} =
\begin{bmatrix}
  b_{1,0} z^{r(1,0)} & a_{1,1} z^{r(1,1)} & a_{1,k} z^{r(1,k)} & \cdots \\
  \vdots & \vdots & \vdots & \vdots \\
  b_{n-1,0} z^{r(n-1,0)} & a_{n,1} z^{r(n,1)} & a_{n,k} z^{r(n,k)} & \cdots \\
  b_{n,0} z^{r(n,0)} & 1 & 1 & \cdots 
\end{bmatrix}
$$

The following two theorems shed some light on what happens in general.

**Theorem 7.** The correspondence between a C-n-fraction and an $n$-tuple of formal power series has the following properties:

(a) To each $n$-tuple there corresponds a C-n-fraction.

(b) To each C-n-fraction without zeros, except those that arise from interruptions, there corresponds an $n$-tuple of formal power series in the following way:

1. If the C-n-fraction terminates at index $k$, then

$$
  f^{(j)}_0(z) = A^{(j)}_k(z)/A^{(0)}_k(z) \quad (1 \leq j \leq n),
$$

and the order in $z$ of the difference between these functions and the $m$-th approximant of the C-n-fraction is increasing in $m$, thus at least $O(z^{m+1})$.

2. If the C-n-fraction does not terminate, then

$$
  f^{(j)}_0(z) - A^{(j)}_k(z)/A^{(0)}_k(z) = O(z^{\sigma(k)+1}) \quad (1 \leq j \leq n),
$$

with $\sigma(k)$ monotonically increasing, thus at least $\sigma(k) \geq k$.

3. If the C-n-fraction for $f^{(1)}_0$, ..., $f^{(n)}_0$ has interruptions of total order $k$, then
there exist at least \( k \) linearly independent relations
\[
P_0(z) + P_1(z)f_0^{(1)}(z) + \ldots + P_n(z)f_0^{(n)}(z) \equiv 0,
\]
with polynomial coefficients. \( \blacksquare \)

**Theorem 8.** Consider a C-n-fraction connected with an n-tuple of formal power series.

(A). The following statements are equivalent:
(a) The C-n-fraction terminates.
(b) The C-n-fraction has nan interruptions of total order \( n \).
(c) There exist \( n \) linearly independent relations with polynomial coefficients as in Theorem 7 (b), (3).
(d) The n-tuple of functions consists of \( n \) rational functions.

B. If \( 1, f_0^{(1)}, \ldots, f_0^{(n)} \) are linearly independent over \( \mathbb{Q}[z] \), the C-n-fraction algorithm has no interruptions (and thus does not terminate). \( \blacksquare \)

Before turning towards convergence matters, first a diagram — in which also a special type of C-n-fractions is introduced — and some examples (for detailed information the reader is referred to [26], [30], [31]). Introduce the following notations

- \( F_n^\mathbb{R} \): n-tuples of formal power series over \( \mathbb{R} \);
- \( F_n^\mathbb{Q} \subseteq F_n \): 1, \( f_0^{(1)}, \ldots, f_0^{(n)} \) are linearly independent over the polynomials over \( \mathbb{Q} \);
- \( C_n \): non-terminating C-n-fractions with all coefficients different from zero;
- \( C_n^{\mathbb{R}} \subseteq C_n \): C-n-fractions with \( b_{j,0} = 1, a_{j,k} \neq 0, r(j, k) = \min(n+1-j, k) \);
- \( \phi \): the construction of a C-n-fraction from an n-tuple of functions;
- \( \chi \): the construction of an n-tuple of functions from a C-n-fraction.

We then have the following inclusion and mapping relations:

\[
\begin{align*}
\chi \circ \phi |_{F_n^\mathbb{R}} & \subset F_n^{\mathbb{Q}} \subset F_n^\mathbb{R} \\
\phi \circ \chi |_{C_n^{\mathbb{R}}} & \subset C_n^{\mathbb{Q}} \subset C_n
\end{align*}
\]

with \( \chi \phi |_{F_n^{\mathbb{R}}} = \text{id} |_{F_n^{\mathbb{R}}} \), \( \phi \chi |_{C_n^{\mathbb{R}}} = \text{id} |_{C_n^{\mathbb{R}}} \).

An interesting question now surfaces immediately: does the absence of an interruption in the C-n-fraction algorithm automatically imply the absence of a dependency relation as introduced in Theorem 7 (c), (3)?

For the Euclidean algorithm for \( n \) real numbers (the Jacobi–Perron algorithm) the answer is as follows (cf. [166]): \( n = 1 \): yes; \( n = 2 \): \(?\); \( n \geq 3 \): \( \text{no.} \)

Choosing the construction method in the context of non-archimedean valuations and a special set of “integers” (replace the polynomials in \( z \) by something a little different), the number of relations is always equal to the number of interruptions (cf. [165]). In the context of the straightforward
generalisation of a C-fraction, however, we have for C-n-fractions (cf. [31]):

\[ n = 1: \text{yes (trivially)}; \quad n \geq 2: \text{no.} \]

Some examples of these special n-tuples are given in the following theorem.

**Theorem 9.** In the sequel we assume that n is an integer with \( n \geq 2 \).

A. Let \( g \) be the unique formal power series with constant term equal to 1
that satisfies

\[ Y^n + (z - 1)Y^{n-1} + z(z - 2)Y^{n-2} + z^2(z - 3)Y^{n-3} + \ldots \]
\[ \ldots + z^{n-2}(z - (n - 1))Y + z^n \equiv 0, \]

(with \( a \in \mathbb{C} \setminus \{0\}, r \geq 1 \)) and define the n-tuple of functions as follows

\[ f_0^{(n)} = g, \quad f_0^{(n+1-j)} = g^j - (g^{j-1} + zg^{j-2} + z^2g^{j-3} + \ldots + z^{j-2}g) \quad (2 \leq j \leq n), \]

then we have

1. There exists precisely one relation over the polynomials, given by

\[ \sum_{j=1}^{n} z^{j-1} f_0^{(j)}(z) + z^n \equiv 0. \]

2. The C-n-fraction has the form

\[
\begin{bmatrix}
  z^{n+1} & z^{n+1} & \ldots \\
-(n-1)z^{n-1} & -(n-1)z^{n-1} & \ldots \\
  z^{n-2} & z^{n-2} & \ldots \\
  \ldots & \ldots & \ldots \\
  z & z & \ldots \\
  1 & 1 & \ldots
\end{bmatrix}.
\]

B. If \( f \) is the unique formal power series with constant term equal to 1 that
satisfies

\[ Y^k - Y^{k-1} - az^r \equiv 0 \quad (k \in \mathcal{N} \setminus \{1\}, r \in \mathcal{N}, a \in \mathbb{C} \setminus \{0\}), \]

then the n-tuple \( f, f^2, \ldots, f^n \) has a C-n-fraction with the properties:

1. For \( n = k - 1 \) there are no interruptions, nor does there exist a relation
with polynomial coefficients for \( 1, f, f^2, \ldots, f^n \).

2. For \( n \geq k \) there are interruptions with total order \( n-k+1 \) and there
exist \( n-k+1 \) linearly independent relations with polynomial coefficients for \( 1, f, f^2, \ldots, f^n \).

The form of the C-n-fraction is best described by giving the entries in an
upwards slanting (under 45°) "diagonal" starting with the top-entry in the first
column (which in (10) has second index 0). Thus the first "diagonal" has two
entries, the second three and so on, till we reach the "diagonal" that starts at
the bottom entry in the first column and which has \( n+1 \) entries, after that
the "diagonals" all start on the bottomline of the array and have \( n+1 \) entries.
(a) For \( n = k - 1 \) the entries on “diagonal” number \( j \) \((1 \leq j \leq n - 1)\) are the monomials that arise from expanding \((1 + az)^j\) using the binomial theorem; after that they are the monomials that arise from expanding \((1 + az)^{k-1}\).

(b) For \( n \geq k \) the entries on “diagonal” number \( j \) \((1 \leq j \leq n - 1)\) are the monomials that arise from expanding \((1 + az)^j\) using the binomial theorem; after that they are the monomials that arise from expanding \((1 + az)^{k-1}\), supplemented by zeros.

There are, of course, other types of examples with \(n\)-tuples showing different behaviour — for many readers a matter of academic interest only maybe — and before looking into convergence just a final example.

Let \( g \) be the unique formal power series with constant term equal to \(1\) that satisfies
\[
g^2 - (1 + 2\delta z^g)g + 2\delta^2 z^{2g} \equiv 0 \quad (\alpha \in \mathcal{N}^+, \delta \in \mathbb{C} \setminus \{0\}),
\]
and define the functions \( f^{(1)}_g, f^{(2)}_g, f^{(3)}_g \) by
\[
f^{(1)}_g = 1 + \frac{az^g}{g} \quad (\beta \in \mathcal{N}, a \in \mathbb{C} \setminus \{0\}), \quad f^{(2)}_g = (f^{(1)}_g)^2, \quad f^{(3)}_g = (f^{(1)}_g)^3.
\]

Then there are two linearly independent relations with polynomial coefficients for \(1, f^{(1)}_g, f^{(2)}_g, f^{(3)}_g\) and the C-3-fraction for \(f^{(1)}_g, f^{(2)}_g, f^{(3)}_g\) has no interruptions and is of the form:
\[
\begin{bmatrix}
az^g & a^2 z^{2g} & a^3 z^{3g} & -4\delta^4 z^{4g} & \ldots & -4\delta^4 z^{4g} & \ldots \\
1 & 2az^g & 3a^2 z^{2g} & 2\delta^2 z^{2g} & 2\delta^2 z^{2g} & \ldots \\
1 & 3az^g & 2\delta z^{g} & 2\delta z^{g} & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots 
\end{bmatrix}
\]

For the remaining part of this section, we will consider C-n-fractions that show the same “minimal”-degree behaviour as regular C-n-fractions (i.e. elements of \( C^n_{\mathbb{C}} \)), but where only the top- and bottom-line in the array (10) have to be different from zero:
\[
\begin{align*}
& r(j, 0) = 0 \quad (1 \leq j \leq \eta), \quad r(j, k) = \min(n + 1 - j, k) \quad (1 \leq j \leq \eta, \ k \geq 1), \\
& b_{j, 0} \neq 0 \quad (1 \leq j \leq \eta), \quad b_k \neq 0, \quad a_{1, k} \neq 0 \quad (k \geq 1).
\end{align*}
\]

Furthermore we need a special real number \( \varepsilon = \varepsilon_n \) defined by
\[
\varepsilon^n + 2^{-1} \varepsilon^{n-1} + 2^{-2} \varepsilon^{n-2} + \ldots + 2^{-n+1} \varepsilon - 2^{-n-1} \equiv 0, \quad 1/6 < \varepsilon \leq 1/4
\]
(simple analysis shows that \( \varepsilon_1 = 1/4, \varepsilon_n \) monotonically decreasing, \( \varepsilon_n \to 1/6 \) for \( n \to \infty \)).

Then we have the following convergence results:

**Theorem 10.** Let \( \{A^{(j)}_n(z)/A^{(j)}_n(z), \ 1 \leq j \leq \eta, \ k \geq 0 \} \) be the approximant \(n\)-tuples of a C-n-fraction satisfying (15), moreover
\[
a_j = \sup_{k \geq 2} |a_{j, k}| < \infty \quad (1 \leq j \leq \eta).
\]
Then the \( n \) sequences converge to an \( n \)-tuple of analytic functions, uniformly in \( z \) on each compact subset of

\[
\mathcal{D} = \{ z \in \mathbb{C} : |z| < \theta_n \cdot \min_{1 \leq j \leq n} a_j^{-1/(n+1-j)} \}.
\]

(\( a_j = 0 \): omit this term from the minimum.)

**Theorem 11.** Consider a \( C-n \)-fraction as in Theorem 10 with

\[
a_j = \limsup_{k \to \infty} |a_{j,k}| < \infty \quad (1 \leq j \leq n).
\]

Then the \( n \) sequences converge to an \( n \)-tuple of functions which are meromorphic on \( \mathcal{D} \) from (17), uniformly in \( z \) on each compact subset of \( \mathcal{D} \). In the poles of the limit functions, the \( C-n \)-fraction shows a very special divergence behaviour, i.e. there exists at least one value \( r \), \( 1 \leq r \leq n \), such that

\[
\lim_{k \to \infty} A_k^{(0)}(z)/A_k^{(l)}(z_0) = 0, \quad \lim_{k \to \infty} A_k^{(0)}(z)/A_k^{(l)}(z_0) \text{ exists for } j \neq r \quad (1 \leq j \leq n).
\]

**Theorem 12.** Consider the situation of Theorem 10 and let

\[
\lim_{k \to \infty} A_k^{(0)}(z)/A_k^{(l)}(z) = g_j(z) \quad (1 \leq j \leq n) \text{ on } \mathcal{D}.
\]

Then the functions \( g_1, \ldots, g_n \) are analytic on \( \mathcal{D} \), together with the function 1 linearly independent over the polynomials and if we apply the construction \( \chi \) on the \( C-n \)-fraction (i.e. we derive the \( n \)-tuple of formal power series that shows agreement in order as described before) then the Maclaurin series of the functions \( g_1, \ldots, g_n \) are recovered.

**Theorem 13.** Consider the situation of Theorem 11 and let

\[
\lim_{k \to \infty} A_k^{(0)}(z)/A_k^{(l)}(z) = g_j(z) \quad (1 \leq j \leq n) \text{ on } \mathcal{D}, \text{ except for the singularities.}
\]

Then the functions \( g_1, \ldots, g_n \) are meromorphic on \( \mathcal{D} \), analytic at the origin, together with the function 1 linearly independent over the polynomials. Moreover, if we apply the construction \( \chi \) on the \( C-n \)-fraction (i.e. we derive the \( n \)-tuple of formal power series \( f_1, \ldots, f_n \) that shows agreement in order as described before) and if \( \mathcal{D} \) is the domain of meromorphy of these formal power series, then \( \mathcal{D} \subset \mathcal{E} \) and \( f_j \equiv g_j \ (1 \leq j \leq n) \) on \( \mathcal{D} \).

**Remark.** For \( n = 1 \) we recover from Theorems 12 and 13 old results due to E. B. van Vleck [191] and A. Pringsheim [172]. Exploiting the connection between certain walks in the Padé-\( n \)-table and \( C-n \)-fractions, it is possible to derive convergence results for sequences of simultaneous rational approximants. In the case that we do not have a \( C-n \)-fraction, but some other form of the coefficients for the recurrence relation, convergence theorems better suited for that situation can easily be given; after some examples connected
With the previous theorems, one of these will be given (dealing with so-called "limitperiodic n-fractions").

**Example. A.** Consider the pair of functions \( f_1(z) = (1 - z)^{1/2} \), \( f_2(z) = (1 - z)^{1/4} \). They give rise to a regular C-2-fraction with coefficients

\[
b_{1,0} = b_{2,0} = 1,
\]

\[
a_{1,1} = 1,
\]

\[
a_{1,3k+1} = \frac{(k + 1/2)(k + 1/4)}{(3k - 1) \cdot 3k \cdot (3k + 1)} \quad (k \geq 1),
\]

\[
a_{1,2} = 1/8,
\]

\[
a_{1,3k+2} = \frac{(k + 1/2)(k + 1/4)}{3k \cdot (3k + 1) \cdot (3k + 2)} \quad (k \geq 1),
\]

\[
a_{1,3k+3} = \frac{(k + 1/2)(k + 3/4)(k + 1)}{(3k + 1) \cdot (3k + 2) \cdot (3k + 3)} \quad (k \geq 0),
\]

\[
a_{2,1} = 1,
\]

\[
a_{2,3k+1} = -\frac{3k + 3/4}{3(3k + 1)} \quad (k \geq 1),
\]

\[
a_{2,2} = -1/4,
\]

\[
a_{2,3k+2} = -\frac{3k^2 + 9k/4 + 1/2}{(3k + 1)(3k + 2)} \quad (k \geq 1),
\]

\[
a_{2,3k+3} = -\frac{k + 1}{(3k + 2)} \quad (k \geq 0).
\]

Thus \( a_1 = 1/27, a_2 = 1/3 \) and \( \mathcal{D} = \{ z \in \mathbb{C} : |z| < 3(3\sqrt{3} - 1)/4 \} \); the radius is approximately 0.549, still "far away" from the expected unit disk.

B. Let \( n \) real numbers be given: \( \alpha_1, \ldots, \alpha_n \in \mathcal{D} \setminus \{1, \ldots, n-1\} \). Introduce real numbers \( B_k(\beta_1, \ldots, \beta_n) \) by

\[
\prod_{j=1}^n (z + \beta_1) - \prod_{j=1}^n \beta_j = \sum_{k=1}^n B_k(\beta_1, \ldots, \beta_n)(z-k+1)_k.
\]

Again the ascending factorial is used: \( (z-k+1)_k = (z-k+1) \cdots (z-1)z \). Introduce the following \( n \)-tuple of combinations of quotients of hypergeometric functions

\[
f^{(n-j)}(z) = \sum_{k=j}^n \frac{B_k(\alpha_1-j, \ldots, \alpha_n-j)}{\prod_{i=1}^n (\alpha_i-j)_{k+1}} \frac{\binom{\alpha_i+k-j+1}{\alpha_i-k-j+1} z^k}{\binom{\alpha_i+1}{\alpha_i+k-j+1} z^{k-1}} F_n(\alpha_1+k-j+1, \ldots, \alpha_n+k-j+1; z)
\]

\[
(1 \leq j \leq n-1),
\]

\[
f^{(n)}(z) = \frac{\binom{\alpha_1+k}{\alpha_1} z}{\binom{\alpha_1+1}{\alpha_1+k} z}.
\]
Then this \(n\)-tuple has a regular \(C\)-fraction with coefficients

\[
b_{n-k,0} = \frac{B_j(\alpha_1-j, \ldots, \alpha_n-j)}{\prod_{i=1}^{n} (\alpha_i-j)_{k+1}} \quad (1 \leq j \leq n-1), \quad b_{n,0} = 1,
\]

\[
a_{n+1-j,k} = \frac{B_j(\alpha_1+k-j, \ldots, \alpha_n+k-j)}{\prod_{i=1}^{n} (\alpha_i+k-j)_{k+1}} \quad (1 \leq j \leq n, \ k \geq 1).
\]

As \(a_j = 0\) \((1 \leq j \leq n)\), the \(C\)-fraction converges to an \(n\)-tuple of functions, meromorphic on \(\mathcal{C}\) and analytic at the origin.

As a final contribution to the theorems concerning convergence, we turn to the subject of modification of a generalised continued fraction (cf. [40], [41], [118]).

**Theorem 14.** Let the coefficients of an \(n\)-fraction be given by

\[
\begin{bmatrix}
  a_1^{(1)} & a_2^{(1)} & a_3^{(1)} & \ldots \\
  0 & a_1^{(2)} & a_2^{(2)} & \ldots \\
  0 & a_1^{(n)} & a_2^{(n)} & \ldots \\
  0 & b_1 & b_2 & b_n \\
\end{bmatrix}
\]

Assume that this \(n\)-fraction is limit-periodic, i.e.

\[
\lim_{k \to \infty} a^{(j)}_k = a^{(j)} \quad (1 \leq j \leq n), \quad \lim_{k \to \infty} b_k = b,
\]

and that the zeros \(z_1, \ldots, z_{n+1}\) of the “auxiliary equation”

\[
z^{n+1} = b_n z^n + a^{(n)} z^{n-1} + \ldots + a^{(2)} z + a^{(1)}
\]

are simple and ordered by the value of their index: \(|z_1| > |z_2| > \ldots > |z_{n+1}|\).

Then the following holds.

A. The \(n\)-fraction converges in \(\mathcal{C}^n = (\mathcal{C} \cup \{\infty\})^n\).

B. If all limits are finite — say \(\xi^{(1)}_0, \ldots, \xi^{(n)}_0\) — and moreover \(|z_{n+1}| > 0\), then all the tails of the \(n\)-fraction converge to finite limits also, say

\[
\begin{bmatrix}
  \xi^{(1)}_k \\
  \vdots \\
  \xi^{(n)}_k \\
\end{bmatrix}
= \begin{bmatrix}
  a_{k+1}^{(1)} & a_{k+2}^{(1)} & \ldots & a_m^{(1)} & \ldots \\
  0 & a_{k+1}^{(2)} & a_{k+2}^{(2)} & \ldots & a_m^{(2)} & \ldots \\
  0 & a_{k+1}^{(n)} & a_{k+2}^{(n)} & \ldots & a_m^{(n)} & \ldots \\
  0 & b_{k+1} & b_{k+2} & \ldots & b_m & \ldots \\
\end{bmatrix} \quad (k \geq 1).
\]

Furthermore, the sequences of tail-values also converge

\[
\lim_{k \to \infty} \xi^{(j)}_k = \omega^{(j)} \quad (1 \leq j \leq n), \text{ where } \omega^{(1)} = \frac{a^{(1)}}{z_1}, \omega^{(j)} = \frac{a^{(j)} + \omega^{(j-1)}}{z_1} \quad (1 \leq j \leq n).
\]
Finally, it can be shown that the introduction of these tail-limits into the method of consecutive linear fractional transformations to calculate the approximants of the $n$-fraction, leads to convergence acceleration. To be more precise, introduce for $k \geq 1$:

$$s_k^{(1)}(y_1, \ldots, y_n) = \frac{a_k^{(1)}}{b_k + y_n}$$

$$s_k^{(j)}(y_1, \ldots, y_n) = \frac{a_k^{(j)} + y_{j-1}}{b_k + y_n} \quad (2 \leq j \leq n),$$

and for $1 \leq j \leq n$, $k \geq 1$ the iterated M"obius-transformations:

$$S_k^{(1)}(y_1, \ldots, y_n) = s_k^{(1)}(y_1, \ldots, y_n),$$

$$S_k^{(j)}(y_1, \ldots, y_n) = S_k^{(0)}(S_k^{(1)}(y_1, \ldots, y_n), \ldots, S_k^{(n)}(y_1, \ldots, y_n)) \quad (k \geq 2)$$

Then the approximant $n$-tuples satisfy

$$S_k^{(0)}(0, \ldots, 0) = \frac{A_k^{(0)}}{A_k^{(0)}},$$

and moreover, the convergence acceleration is shown by:

$$\lim_{k \to \infty} \frac{\xi_k^{(0)} - S_k^{(0)}(w^{(1)}, \ldots, w^{(n)})}{\xi_k^{(0)} - S_k^{(0)}(0, \ldots, 0)} = 0. \quad \blacksquare$$

It would take too much time to go into details here, the procedure is straightforward. Consider an $n$-fraction where the coefficients are functions of a complex variable $z$.

Then take for instance the following steps

- check the limit-periodicity,
- check the auxiliary equation,
- the condition of simple zeros, ordered as in the theorem, leads to conditions on $z$,
- check the convergence to finite limits, using Theorem 6,
- translate the conditions on $z$ into a domain (?) in $\mathcal{S}$.

4. Convergence in the Padé-$n$-table

It is out of the question to treat all known convergence results here, therefore we will restrict ourselves mainly to the list of (semi)-normal tables given in Theorem 2, along with information on several "more general" results.

(a) the exponential function system $(e^{\lambda_j z}, 1 \leq j \leq n)$ with $\lambda_j \neq 0$, $\lambda_j \neq \lambda_i$.

Convergence to the $n$-tuple of functions for any sequence $(c_0(k), c_1(k), \ldots, c_n(k))$ with $c(k) = c_0(k) + c_1(k) + \ldots + c_n(k) \to \infty$ for $k \to \infty$. Note that there is no monotonicity condition on the order of approximation. If $\sigma(k) \to \infty$ and $c_j/c_0 \to \omega_j \ (1 \leq j \leq n)$ for $k \to \infty$, then the denominators and numerators
converge separately. All convergence results are uniform on compact subsets of \( C \) (cf. J. Mall, A. J. Goddijn, A. I. Aptekarev).

(b) the binomial function system \( ((1-z)^{\lambda_j}, \ 1 \leq j \leq n) \) with \( \lambda_j \notin \mathcal{A} \), \( \lambda_j - \lambda_{j+1} \notin \mathcal{A} \).

Convergence for \( q_0 \to \infty \), \( q_j \) fixed \( (1 \leq j \leq n) \) on \( |z| < 1 \) (cf. A. J. Goddijn).

(c) Angelesco-systems.

Convergence results by a.o. V. A. Kalyagin, E. M. Nikishin, V. N. Sorokin.

(e) the hypergeometric function system \( \left( _{i}F_{1}(1; \ c; \ z), \ 1 \leq j \leq n \right) \), \( c_j \notin \mathcal{A} \backslash \mathcal{N} \), \( c_i - c_j \notin \mathcal{A} \).

Convergence to the \( n \)-tuple of functions for any sequence \( (q_0(k), q_1(k), \ldots, q_n(k)) \) with \( \sigma(k) = q_0(k) + q_1(k) + \ldots + q_n(k) \to \infty \) for \( k \to \infty \). Note that there is no monotonicity condition on the order of approximation. If \( \sigma(k) \to \infty \) and \( q_j/q_0 \to \omega_j \) \( (1 \leq j \leq n) \) for \( k \to \infty \), then the denominators and numerators converge separately. All convergence results are subject to the condition \( q_0 \geq q_j - 1 \) \( (1 \leq j \leq n) \), and uniform on compact subsets of \( C \); no monotonicity condition (cf. M. G. de Bruin).

(f) the hypergeometric function system \( \left( _{i}F_{1}(1; \ c; \ \lambda_j z), \ 1 \leq j \leq n \right) \) with \( \lambda_j \neq 0 \), \( \lambda_j \neq \lambda_i \).

Convergence to the \( n \)-tuple of functions for any sequence \( (q_0(k), q_1(k), \ldots, q_n(k)) \) with \( \sigma(k) = q_0(k) + q_1(k) + \ldots + q_n(k) \to \infty \) for \( k \to \infty \), subject to the condition \( q_0 \geq q_j - 1 \) \( (1 \leq j \leq n) \). Note that there is no monotonicity condition on the order of approximation (cf. A. I. Aptekarev).

(f) de Montessus de Ballore-type theorems.


(g) diagonal-"near" diagonal sequences under general conditions (capacity-results etc.).


From the literature it appears that there are several methods to prove convergence in the Padé-\( n \)-table:

1. Using asymptotics for the determinants in the explicit forms for the denominators and remainders.
2. Using potential theory and the theory of (generalised) orthogonal polynomials.
3. Using explicit calculation and estimation for denominators and remainder.
4. Using the connection with generalised continued fractions for certain sequences of approximants.

It must be noted, however, that the convergence problem is very difficult, but progress is made in several directions.
5. Concluding remarks

The number of applications of simultaneous rational approximation with common denominator is steadily growing. In this connection the work on time-series (M. Hallin), vector Padé approximants (P. R. Graves-Morris, J. M. Wilkins, J. van Iseghem) must be mentioned. Furthermore applications to “two-point approximation” and “partial Padé approximation” ([39]) are being studied at the moment. The last mentioned application shows great potential for improvement of numerical approximation. Here information on eventual singularities and/or zeros of the functions is used to first “force” a similar behaviour on the rational approximants by prescribing factors in numerator and/or denominator and then imposing conditions on the order of approximation.

We consider the pair of functions

\[ f_1(t) = \frac{\exp t}{1 - 2t} = 1 + 3t + \frac{13}{2} t^2 + \frac{79}{6} t^3 + \ldots, \]

\[ f_2(t) = (1 - t)^{3/2} = 1 - \frac{3}{2} t + \frac{3}{8} t^2 + \frac{1}{16} t^3 + \ldots. \]

In order to imitate both the pole at \( t = 0.5 \) and the limiting value 0 when approaching \( t = 1.0 \) from below along the real axis — we introduce the polynomials

\[ v_1(t) = 1, \quad v_2(t) = (t - 1)(t - 2), \quad w(t) = t - 2, \]

which leads for instance to the approximation problem (for details on the degree restrictions cf. [39])

\[ \tilde{p}_j(t) \tilde{q}_j(t) - \tilde{q}(t) \tilde{w}(t) f_j(t) = O(t^3) \quad (j = 1, 2), \]

where the approximants we want to calculate are nothing else but \( \tilde{p}_j(t) \tilde{v}_j(t) / \tilde{q}(t) \tilde{w}(t) \). Dividing out the auxiliary polynomials, the problem reduces to

\[ \tilde{p}_1(t) - \tilde{q}(t) \exp(t) = O(t^3), \quad \tilde{p}_2(t) - \tilde{q}(t) \sqrt{1 - t} = O(t^3). \]

Calculating the approximants we find

\[ \tilde{p}_1(t) \tilde{v}_1(t) / \tilde{q}(t) \tilde{w}(t) = (1 + \frac{1}{12} t)/(1 - \frac{1}{12} t - \frac{1}{12} t^2) (1 - 2t), \]

\[ \tilde{p}_2(t) \tilde{v}_2(t) / \tilde{q}(t) \tilde{w}(t) = (1 - \frac{1}{12} t)/(1 - \frac{1}{12} t - \frac{1}{12} t^2). \]

The approximants to \( f_j \) have simple poles at \( t = (-5 \pm \sqrt{73})/2 \), i.e. at \( t = 1.772 \) and \( t = -6.772 \), moreover the approximant to \( f_1 \) also has a simple pole at \( t = 0.5 \), while that to \( f_2 \) has a zero at \( t = 1.0 \). We compare this with the \( (0, 1, 1) \) system of German polynomials for \{\( f_1, f_2 \)\}; we find

\[ P_1(0, 1, 1; t) = 1 + \frac{32}{3} t, \quad P_2(0, 1, 1; t) = 1 - \frac{163}{30} t, \]

\[ P_0(0, 1, 1; t) = 1 - \frac{49}{15} t - \frac{23}{15} t^2. \]
The (0, 1, 1) approximants have both the same simple poles at \( t = (-49 \pm \sqrt{14929})/174 \), i.e. at \( t = -0.983 \) and \( t = 0.421 \). Plotting the Euclidean distance in 2-space between the pair of functions and their pair of approximants, the partial Padé approximants show a superior numerical behaviour. This will be a matter of continuing research.

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