CONTRACTIVE OPERATORS OF CERTAIN SPACES

BY

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Introduction. Let $X$ be a measure (topological) space and $A(X)$ a normed algebra of functions on $X$. An important problem in analysis is to characterize homomorphisms of $A(X)$. Saeki [5] considered the contractive homomorphisms of the tensor algebra $C(X) \hat{\otimes} C(X)$, where $C(X)$ is the space of continuous functions on the compact space $X$. Cohen [1], Wood [7], Greenleaf [2], and Rigelhof [4] studied norm decreasing homomorphisms of the group algebra $L^1(G)$ and the measure algebra $M(G)$ for a locally compact abelian group $G$.

In this paper* we study norm decreasing operators on $L^2(I, m) \hat{\otimes} L^2(I, m)$ which are of the form $U(\varphi) = \varphi \circ F$ for all trace class functions $\varphi$ ($F$ is a measurable map on $I \times I$). We prove that if

$$||U(\varphi)||_{\text{Tr}} = ||\varphi \circ F||_{\text{Tr}} \leq ||\varphi||_{\text{Tr}},$$

then $F$ is essentially of the form $(F_1, F_2)$, where $F_1$ and $F_2$ are measure-preserving maps on $I$; that is

$$F(x, y) = (F_1(x), F_2(y)) \quad \text{or} \quad F(x, y) = (F_1(y), F_2(x)).$$

Contractive maps of the trace class operators. Let $I$ denote the unit interval with the usual Lebesgue measure $m$ and let $L^2(I, m) \hat{\otimes} L^2(I, m)$ be the complete projective tensor product of $L^2(I, m)$ with itself. It is well known [6] that $L^2(I, m) \hat{\otimes} L^2(I, m)$ is isometrically isomorphic to the space of the trace class operators. If $\psi \in L^2(I, m) \hat{\otimes} L^2(I, m)$, then $||\psi||_{\text{Tr}}$ denotes the trace class norm of $\psi$, and $||\psi||_{\text{HS}}$ its Hilbert-Schmidt norm.

Theorem. Let $F: I \times I \to I \times I$ be a measurable map and let

$$U: L^2(I, m) \hat{\otimes} L^2(I, m) \to L^2(I, m) \hat{\otimes} L^2(I, m)$$

be a contractive operator defined by $U(\varphi) = \varphi \circ F$. Then $F$ is essentially of the form $(F_1, F_2)$, where $F_1$ and $F_2$ are measure-preserving maps on $I$.

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One has to remark here that if \( F = (F_1, F_2) \) and \( F_1, F_2 \) are measure-preserving maps on \( I \), then the operator \( U \) on the trace class functions defined by \( U(\varphi) = \varphi \circ (F_1 \otimes F_2) \) is contractive.

**Proof of the Theorem.** Since the proof is long, we prove the claim in four steps.

**Step I.** The mapping \( F \) is measure preserving.

Let \( X_1 \) and \( X_2 \) be any two disjoint sets in \( I \) such that \( I = X_1 \cup X_2 \). If \( Y_1 \) and \( Y_2 \) is a similar pair of sets in \( I \), then we put \( 1_{X_i \times Y_j} \) to denote the characteristic function of \( X_i \times Y_j \), \( 1 \leq i, j \leq 2 \). From the definition of the Hilbert-Schmidt norm we obtain

\[
[m(F^{-1}(X_i \times Y_j))]^{1/2} = \|1_{X_i \times Y_j} \circ F\|_{\text{HS}} \leq \|1_{X_i \times Y_j} \circ F\|_{\text{Tr}} \leq |m(X_i \times Y_j)|^{1/2}.
\]

Now we have

\[
1 = \sum_{i,j=1}^{2} m(F^{-1}(X_i \times Y_j)) \leq \sum_{i,j=1}^{2} m(X_i \times Y_j) = 1.
\]

Hence \( m(F^{-1}(X_i \times Y_j)) = m(X_i \times Y_j), \ 1 \leq i, j \leq 2 \). Consequently, \( F \) preserves the measure of rectangles. Since the \( \sigma \)-algebra of Lebesgue measurable sets is the completion of the smallest \( \sigma \)-algebra containing the rectangles, \( F \) is a measure-preserving map on \( (I \times I, m \times m) \).

**Step II.** The operator \( U \) preserves atoms in \( L^2(I, m) \otimes L^2(I, m) \).

Let \( f \otimes g \in L^2(I, m) \otimes L^2(I, m) \). If \( \|f \otimes g\|_2 \) denotes the norm of \( f \otimes g \) as an element of \( L^2(I \times I, m \otimes m) \), then \( \|f \otimes g\|_2 = \|f \otimes g\|_{\text{Tr}} = \|f\|_2 \|g\|_2 \). Since \( F \) is measure preserving (step I), one can prove that \( U \) is an isometry on \( L^2(I \times I, m \otimes m) \). Hence

\[
\|f \otimes g\|_{\text{HS}} = \|(f \otimes g) \circ F\|_{\text{HS}} \leq \|(f \otimes g) \circ F\|_{\text{Tr}} \leq \|f \otimes g\|_{\text{Tr}} = \|f \otimes g\|_2.
\]

Therefore, \( \|(f \otimes g) \circ F\|_{\text{HS}} = \|(f \otimes g) \circ F\|_{\text{Tr}} \), which is possible only if \( (f \otimes g) \circ F \) is of rank one. That is, \( (f \otimes g) \circ F = u \otimes v \) for some atom \( u \otimes v \) in \( L^2(I, m) \otimes L^2(I, m) \).

**Step III.** Construction of \( F_1 \) and \( F_2 \).

Let \( i: I \to I \) be the identity map \( i(x) = x \), and let \( \pi_1, \pi_2: I \times I \to I \) be the first and the second projections, respectively. Set \( F_1 = \pi_1 \circ F \) and \( F_2 = \pi_2 \circ F \). Then \( F(x, y) = (F_1(x, y), F_2(x, y)) \). Consider the map

\[
i \otimes 1: I \times I \to I \times I,
\]

where \( 1 \) is the constant function with range \( \{1\} \). Step II implies that \( (i \otimes 1) \circ F = \alpha_1 \otimes \alpha_2 \) for some \( \alpha_1 \otimes \alpha_2 \) in \( L^2(I, m) \otimes L^2(I, m) \). Hence \( F_1(x, y) = \alpha_1(x) \cdot \alpha_2(y) \). Similarly, \( F_2(x, y) = \beta_1(x) \cdot \beta_2(y) \) for some \( \beta_1 \otimes \beta_2 \) in \( L^2(I, m) \otimes L^2(I, m) \).

**Step IV.** Each of the functions \( F_1 \) and \( F_2 \) depends on one of the variables \( x \) and \( y \), but not on both of them.
For any function \( \psi \in L^2(I, m) \otimes L^2(I, m) \), set
\[
m(\psi) = \int_I \int_I \psi(x, y) \, dx \, dy.
\]
From step I it follows that
(1) \[ m(\psi) = m(U(\psi)). \]
Now, if \( \varphi \) is an atom in \( L^2(I, m) \otimes L^2(I, m) \), we put
\[
m_1(\varphi) = \int_I \varphi(x, y) \, dy \quad \text{and} \quad m_2(\varphi) = \int_I \varphi(x, y) \, dx.
\]
Hence we can write
(2) \[ m(\varphi) \cdot \varphi = m_1(\varphi) \otimes m_2(\varphi). \]
Step II together with (1) implies
\[
m(\varphi) \cdot U(\varphi) = m_1(U(\varphi)) \otimes m_2(U(\varphi)).
\]
Therefore, if \( m(\varphi) = 0 \), then either \( m_1(U(\varphi)) = 0 \) or \( m_2(U(\varphi)) = 0 \). Now, take \( \varphi = f \otimes 1 \) and write \( U(f) \) instead of \( U(f \otimes 1) \). Set
\[
V_j = \{ f \mid m_j(U(f - m(f) \cdot 1)) = 0 \},
\]
where \( m(f) \) denotes \( m(f \otimes 1) \). Since for any \( f \in L^2(I, m) \) we have
\[
m(f - m(f) \cdot 1) = m(f) - m(f) = 0,
\]
it follows that for any \( f \in L^2(I, m) \) either \( m_1(U(f - m(f) \cdot 1)) = 0 \) or \( m_2(U(f - m(f) \cdot 1)) = 0 \). Hence \( V_1 \) and \( V_2 \) are closed subspaces of \( L^2(I, m) \) such that \( L^2(I, m) = V_1 \cup V_2 \). In this case, as well known, either \( V_1 = L^2(I, m) \) or \( V_2 = L^2(I, m) \). That is, there exists \( j = 1 \) or \( j = 2 \) such that
\[
m_j(U(f - m(f) \cdot 1)) = 0
\]
for all \( f \in L^2(I, m) \). Without loss of generality, we can assume that \( j = 1 \). Thus \( m_1(U(f)) = m(f) \cdot 1 \). Relations (1) and (2) then imply
\[
m(f) \cdot U(f) = m(f) \cdot (1 \otimes m_2(U(f))).
\]
Hence, if \( m(f) \neq 0 \), we obtain
\[
U(f) = U(f \otimes 1) = 1 \otimes m_2(U(f)).
\]
On the other hand, we get
\[
U(f \otimes 1) = (f \otimes 1) \circ (F_1, F_2) = (f \circ F_1) \otimes 1.
\]
Consequently, using step III, for any \( (x, y) \in I \times I \) we have
\[
m_2(U(f))(y) = ((f \otimes 1) \circ F)(x, y) = (f \circ F_1)(x, y) = f(\alpha_1(x) \alpha_2(y)).
\]
Hence $\alpha_1$ is a constant. In case $j = 2$, we can see that $\alpha_2$ is a constant. This shows that the function $F_1$ depends either on the first coordinate or on the second coordinate but not on both of them. One can prove the same thing for $F_2$ considering the atom $1 \otimes f$. We conclude that $F$ takes one of the following forms:

\[ F = (\alpha_1, \beta_1), \quad F = (\alpha_1, \beta_2), \]
\[ F = (\alpha_2, \beta_1), \quad F = (\alpha_2, \beta_2). \]

Finally, we have to show that $\alpha_1$, $\alpha_2$, $\beta_1$, and $\beta_2$ are all measure-preserving maps on $I$. We prove the claim only for $\alpha_1$. Now, let $E$ be any set in $I$. Put $f = 1_E$ and consider $f \otimes 1 = 1_E \otimes 1$. Since $F$ is measure preserving, we have

\[ [m(E)]^{1/2} = \|1_E \otimes 1\|_2 = \|(1_E \otimes 1) \circ F\|_2 = \|1_E(\alpha_1)\|_2 = [m(\alpha_1^{-1}(E))]^{1/2}. \]

(Here we are considering $F$ to be of the form $F = (\alpha_1, \beta_1)$.) This completes the proof of the Theorem.

REFERENCES


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