On the existence and uniqueness of the solution of a non-linear functional equation of r-th order

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Abstract. In the paper the functional equation

\[ x(t) = f\{t, x(\beta_1(t)), \ldots, x(\beta_r(t))\} \]

is discussed. An unknown function \( x \) is defined in a metric space and takes the value in another complete metric space. The method of successive approximations is used. Under some assumptions concerning the Lipschitz coefficients of function \( f \) and the functions \( \beta_i \) the existence and uniqueness of solution in the suitable class of functions is established.

In the present note we consider a non-linear functional equation of the form

\[ x(t) = f\{t, x(\beta_1(t)), \ldots, x(\beta_r(t))\} \]  

with an unknown function \( x \) defined in a metric space \((M_1, \rho_1)\) having the range in a complete metric space \((M_2, \rho_2)\). The special cases of equation (1) were considered by several authors: see [1]-[8]. In the general case the existence and uniqueness problem for equation (1) was discussed in [7]. There the comparative method was used and the comparative equation was solved by the method of iteration. This method permits us to establish the general results concerning the problem in question but they are somewhat hidden behind the rather complicated formulas.

In the present note we prove by the Banach fixed-point theorem some results concerning the existence and uniqueness problem for equation (1). This result is more general then the corresponding results established in [1], [4], [5], [6], [8].

1. Let \((M_i, \rho_i), i = 1, 2,\) be arbitrarily fixed metric spaces and let \((M_2, \rho_2)\) be complete. We make

**Assumption A. Suppose that:**

1° \( f: M_1 \times M_i \rightarrow M_2, \beta_i: M_1 \rightarrow M_1, i = 1, 2, \ldots, r, \)

where \( r \) is a fixed positive integer number (the case \( r = +\infty \) is not excluded).
2° there exist \( t_0 \in M_1 \), a function \( \varphi_0: M_1 \rightarrow M_2 \), \( p, H \in \mathbb{R}_+ \) such that

\[
\varrho_2(\varphi(\varphi_0(t), \varphi_0(t)) \leq H \varrho_2^p(t_0, t), \quad t \in M_1,
\]

where \( \varphi(\varphi_0(t)) \) denotes the right-hand side of equation (1).

3° there exist functions \( l_i: M_1 \rightarrow \mathbb{R}_+ \) such that

\[
\varrho_2(f(t, x_1, \ldots, x_r), f(t, \bar{x}_1, \ldots, \bar{x}_r)) \leq \sum_{i=1}^r l_i(t) \varrho_2(x_i; \bar{x}_i)
\]

for any \( t \in M_1 \) and \( x_i, \bar{x}_i \in M_2, \ i = 1, 2, \ldots, r. \)

Using the notation introduced, we define the set \( V_p \) of functions \( \varphi \) defined in \( M_1 \) with range in \( M_2 \) such that \( \varphi \in V_p \) iff there exist \( c_\varphi \in \mathbb{R}_+ \) and

\[
\varrho_2(\varphi(t), \varphi_0(t)) \leq c_\varphi \varrho_2^p(t_0, t), \quad t \in M_1.
\]

In \( V_p \) we define the metric \( \varrho \): for \( \varphi, \psi \in V_p \) we put

\[
\varrho(\varphi, \psi) = \inf \{ c: \varrho_2(\varphi(t), \psi(t)) \leq c \varrho_2^p(t_0, t), t \in M_1 \}.
\]

It is clear that \( (V_p, \varrho) \) is a complete metric space.

2. Now we have

**Theorem A.** If Assumption A is satisfied and

\[
\sup_{t \in M_2 \setminus t_0} \sum_{i=1}^r l_i(t) \varrho_2^p(t_0, \beta_i(t)) \varrho_2^p(t_0, t) = a < 1,
\]

then there exists in \( V_p \) a unique solution \( \bar{x} \) of equation (1) and it can be obtained by the method of successive approximations.

**Proof.** First we observe that \( \varphi(V_p) \subset V_p \). Indeed by (2), (3) and the definition of \( V_p \) we have for \( \varphi \in V_p \)

\[
\varrho_2(\varphi_0(t), F(\varphi)(t)) \leq \varrho_2(\varphi_0(t), F(\varphi_0)(t)) + \varrho_2(F(\varphi_0)(t), F(\varphi)(t))
\]

\[
\leq H \varrho_2^p(t_0, t) + \sum_{i=1}^r l_i(t) \varrho_2(\varphi_0(\beta_i(t)), \varphi_0(\beta_i(t)))
\]

\[
\leq H \varrho_2^p(t_0, t) + \sum_{i=1}^r l_i(t) c_\varphi \varrho_2^p(t_0, \beta_i(t))
\]

\[
\leq H \varrho_2^p(t_0, t) + \left( \sup_{t \in M_2 \setminus t_0} \sum_{i=1}^r l_i(t) \varrho_2^p(t_0, \beta_i(t)) \right) \times \varrho_2^p(t_0, t)
\]

\[
\leq (H + ac) \varrho_2^p(t_0, t).
\]
Further we prove that $F$ is a contraction in $V_p$. For $x, y \in V_p$ there exist $c \in \mathbb{R}_+$ such that

$$e_2(x(t), y(t)) \leq c e_1^p(t_0, t), \quad t \in M_1;$$

thus for any such $c$ we get

$$e_2(F(x)(t), F(y)(t)) \leq \sum_{i=1}^{r} l_i(t) e_2(x(\beta_i(t)), y(\beta_i(t))) \leq c \sum_{i=1}^{r} l_i(t) e_1^p(t_0, \beta_i(t))$$

$$\leq \left( \sup_{t \in M \setminus \{t_0\}} \sum_{i=1}^{r} l_i(t) \frac{e_1^p(t_0, \beta_i(t))}{e_1^p(t_0, t)} \right) c e_1^p(t_0, t)$$

$$= ac e_1^p(t_0, t);$$

hence in view of the definition of the metric in $V_p$ we obtain

$$e(F(x), F(y)) \leq a e(x, y);$$

now the assertion of the theorem is implied by the Banach fixed point theorem.

3. In order to formulate another theorem we introduce

**Assumption B.** Suppose that Assumption A is satisfied and there exist constants $\mu \in \mathbb{R}, \nu, L_i, B_i \in \mathbb{R}_+$, $i = 1, \ldots, r$, such that

(5) $l_i(t) \leq L_i e_1^p(t_0, t), \quad t \in M_1, \ i = 1, \ldots, r,$

(6) $e_1(t_0, \beta_i(t)) \leq B_i e_1^p(t_0, t), \quad t \in M_1, \ i = 1, \ldots, r.$

From Theorem A we infer

**Theorem B.** If Assumption B is satisfied, $\mu + \nu p = p$ and

(7) $\sum_{i=1}^{r} L_i B_i^p < 1,$

then there exists in $V_p$ a unique solution $\bar{x}$ of equation (1).

**Proof.** By assumptions (5)–(7) we have

$$\sup_{t \in M \setminus \{t_0\}} \sum_{i=1}^{r} l_i(t) \frac{e_1^p(t_0, \beta_i(t))}{e_1^p(t_0, t)} \leq \sup_{t \in M \setminus \{t_0\}} \sum_{i=1}^{r} L_i e_1^p(t_0, t) \frac{B_i e_1^p(t_0, t)}{e_1^p(t_0, t)}$$

$$= \sum_{i=1}^{r} L_i B_i^p < 1;$$

now the assertion of the theorem results by Theorem A.
References


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