AN AXIOM SYSTEM
FOR THE CLASS OF GROUPS OF DILATATIONS
IN FANO–PAPPIAN AFFINE PLANES

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Introduction. In this paper an axiom system for the class of Fano–Pappian plane dilatation groups will be given. It will be shown how to find an axiom system for classes of dilatation groups of Fano–Desarguesian affine planes, affine spaces, and Fano–Pappian affine spaces (dim $\geq 3$).

We base on affine geometry with parallelity, either two-dimensional with Szmielew's axioms or of higher dimension with Kusak's axioms (see [2] and [1]). In both of these geometries we assume the axiom of Fano or at least the axiom of Desargues.

In models for such geometries we can define central symmetry as a dilatation which is simultaneously an involution. We know from the Fano axiom and the property of rigidness that a central symmetry has exactly one fixed point. From the Desargues theorem it follows that around each point there exists a dilatation which is an involution. We can therefore identify involutions of a dilatation group with their fixed points. This enables us to create axiom systems for classes of dilatation groups of appropriate affine planes or spaces.

1. Dilatation group of affine plane. By a plane affine geometry we shall understand the theory based on the following Szmielew's axioms:

S1.0. $ab \parallel ba$.
S1.1. $ab \parallel cc$.
S1.2. $a \neq b \land ab \parallel pq \land ab \parallel rs \rightarrow pq \parallel rs$.
S1.3. $ab \parallel ac \rightarrow ba \parallel bc$.
S1.4. $(\exists a, b, c) \sim ab \parallel ac$.
S1.5. $(\forall a, b, p) \exists q \ (ab \parallel pq \land p \neq q)$.
S1.6. $\sim ab \parallel cd \rightarrow \exists p \ (pa \parallel pb \land pc \parallel pd)$.
Let us denote this theory by $\text{Af}_2$; every its model will be called an affine plane.

Let us denote by $F$ the sentence
\[ ab \parallel cd \land ac \parallel bd \land ad \parallel bc \rightarrow ab \parallel ac, \]
by $D$ the sentence
\[ \sim ab \parallel ap \land \sim ab \parallel ar \land o \neq a, b, p, q, r, s \land oa \parallel ob \land op \parallel oq \land or \parallel os \land ap \parallel bq \land ar \parallel bs \rightarrow pr \parallel qs, \]
and by $P$ the sentence
\[ ab \parallel ac \land de \parallel df \land ae \parallel bd \land hf \parallel ce \rightarrow af \parallel cd. \]

If we enrich the axioms of $\text{Af}_2$ by the sentences $F, D$ or $P$, we shall obtain Fanoian, Desarguesian or Pappian affine geometry, respectively.

Let $A = \langle S, \parallel \rangle$ be a model of $\text{Af}_2$. The set of affine plane dilatations, i.e., the set of all bijections $f$ of the plane $S$ satisfying the condition $ab \parallel f(a)f(b)$ for all $a, b \in S$, will be denoted by $\text{Dil}(A)$.

We recall some basic properties of $\text{Dil}(A)$.

**Theorem 1.1.** $\text{Dil}(A)$ forms a group of transformations.

**Theorem 1.2** (rigidity). Let $f \in \text{Dil}(A)$ and $a, b \in S$. If $a \neq b$ and $f(a) = a, f(b) = b$, then $f = \text{Id}$.

**Corollary 1.1.** Every non-identical dilatation has at most one fixed point.

Let $A = \langle S, \parallel \rangle$ be a Fano–Desarguesian affine plane.

**Theorem 1.3** (homogeneity). We have
\[ ab \parallel cd \land a \neq b, c \neq d \rightarrow \exists f (f \in \text{Dil}(A) \land f(a) = c \land f(b) = d). \]

For involutions of such planes the following theorems hold:

**Theorem 1.4.** $f \in \text{Dil}(A) \land f^2 = \text{Id} \neq f \rightarrow \exists! p (p \in S \land f(p) = p)$.

**Theorem 1.5.** $\forall p \in S \exists! f (f \in \text{Dil}(A) \land f^2 = \text{Id} \neq f \land f(p) = p)$.

**Definition 1.1.** Let $\sigma_p$ be a transformation from $\text{Dil}(A)$ such that $\sigma_p^2 = \text{Id} \neq \sigma_p$ and $\sigma_p(p) = p$.

**Corollary 1.2.** $f \in \text{Dil}(A) \rightarrow (f^2 = \text{Id} \neq f \Leftrightarrow \exists p (p \in S \land f = \sigma_p))$.

**Theorem 1.6.** If $f \in \text{Dil}(A)$ and $p \in S$, then $f \sigma_p f^{-1} = \sigma_{f(p)}$.

Let $\text{Tr}(A)$ be the set of dilatations without fixed points and identity, and $J_p(A)$ the set of dilatations with a fixed point $p$. Notice that $\sigma_p \in J_p(A)$.

2. Dilatation group theory. We shall give here the axiom system of Fano–Pappian affine plane dilatation groups.

Let $\text{Dil}_2$ be a theory based on the following axioms:

G2.1. $a1 = 1a = a$.

G2.2. $\forall a \exists x (xa = ax = 1)$. 
G2.3. \((ab)c = a(bc)\).

G2.4. \((a^2 = 1 \land b^2 = 1 \land c^2 = 1)\)
   \[\rightarrow ((abc)^2 = 1 \lor a = 1 \lor b = 1 \lor c = 1).\]

G2.5. \((ab = ba \land ac = ca) \rightarrow (bc = cb \lor a = 1)\).

G2.6. \((\forall a, b, c)((a^2 = b^2 = c^2 = 1 \land a, b, c \neq 1)\)
   \[\rightarrow (\exists d (d \neq 1 \land ad = da \land bd = db))\].

G2.7. \((\exists a, b, c \forall d (a^2 = b^2 = c^2 = 1 \land a, b, c \neq 1 \land a \neq b, c\)
   \land (ad = da \rightarrow db \neq cd))\).

G2.8. \((\forall a, b, c, d)(\exists e, f, g)\left[\left((a^2 = b^2 = c^2 = d^2 = 1 \neq a, b, c, d\right)_\land\right.
   \land (a, b, c) \neq d \neq a, c) \rightarrow ((fa = ef \land fb = df
   \land (ga = ag \land ge = cg \land a = e)) \lor (fa = af \land fc = df)
   \lor (fa = af \land fb = c f))\right]\].

**Theorem 2.1.** Let \(M \models \text{Af}_2 + P + F\). Then \(\text{Dil}(M) \models Dl_2\).

**Proof.** Take \(M\) satisfying \(\text{Af}_2 + P + F\) and consider \(\text{Dil}(M)\).

1° \(\text{Dil}(M) \models G2.1, G2.2\) and \(G2.3\). \(G2.1\)–\(G2.3\) state that the discussed structure forms a group, so they are a consequence of Theorem 1.1.

2° \(\text{Dil}(M) \models G2.4\). Assume \(f, g, h\) are involutions, so from Corollary 1.2 we obtain \(f = \sigma_p, g = \sigma_q, h = \sigma_r\) for some \(p, q, r \in |M|\). Then \(\sigma_p \sigma_q \in \text{Tr}(M)\). Therefore for some \(s\) we have \(\sigma_s \sigma_q \sigma_r = \sigma_s\). Thus \(\sigma_p \sigma_q \sigma_r = \sigma_s\).

3° \(\text{Dil}(M) \models G2.5\). It is easily seen that if \(f \in J_p(M), f \neq \text{Id}\) and \(fg = gf\) for \(g \in \text{Dil}(M)\), then \(g \in J_p(M)\).

Assume \(f \neq \text{Id}\). Consider two cases:

(i) \(f \in J_p(M)\). Then \(g, h \in J_p(M)\), which gives \(gh = hg\) as a consequence of \(P\).

(ii) \(f \in \text{Tr}(M)\). Then \(g, h \in \text{Tr}(M)\), so \(gh = hg\), which is a consequence of \(D\).

4° \(\text{Dil}(M) \models G2.6\). By the assumption there are points \(p, q, r\) such that \(f = \sigma_p, g = \sigma_q, h = \sigma_r\). Take \(j \in \text{Dil}(M)\) such that \(j \neq \text{Id}\) and \(j^2 = jf\) and \(aj = jh\). For such \(j\) we have \(j(p) = p\) and \(j(r) = q\). We search \(k \in \text{Dil}(M)\) such that \(k \neq \text{Id}, k(r) = r, k(q) = p\). Assume that \(p, q, r\) (possibly taking \(f\) for \(k\)). We have \(pr \parallel pq, \text{ so } qr \parallel qp\). The existence of \(k\) follows from Theorem 1.3.

5° \(\text{Dil}(M) \models G2.7\). By S1.4 there exist points \(a, b, c\) such that \(\sim ab \parallel ac\) (thus \(a \neq b, c\) and \(b \neq c\)). Therefore, there is no \(f \in \text{Dil}(M)\) such that \(f(a) = a\) and \(f(b) = c\). We take involutions \(\sigma_a, \sigma_b, \sigma_c\).

6° \(\text{Dil}(M) \models G2.8\). Notice the following fact remains true for every affine plane. For all points \(a, b, c, d\) such that \(\sim ab \parallel ac\) there exists a point \(e\) such that \(ab \parallel de\) and \(ac \parallel ae\). So if we have involutions \(\sigma_a, \sigma_b, \sigma_c, \sigma_d\), then the existence of the dilatations we seek follows from Theorem 1.3.

The following sentences can be proved in \(Dl_2\).
Lemma 2.1. \((ab = ba \land a^2 = 1 \land b^2 = 1) \rightarrow (a = b \lor a = 1 \lor b = 1)\).

Proof. Let \(a, b \neq 1\). Denote \(aba^{-1}\) by \(c\). We have then \(ab = ca\) and \(a \neq 1\). By G2.6 there exists \(f\) such that \(fc = cf\) and \(fb = bf\). From the equality \(ab = ba\) we obtain \(c = b, fb = bf\) and \(fb = af\). This gives \(a = b\).

Lemma 2.2. We have

\[(\forall a, b) \exists c (a^2 = b^2 = 1 \neq a, b \rightarrow ac = cb \land c^2 = 1 \land c \neq 1).\]

Proof. When \(a = b\), for \(c\) take \(a\). Assume \(a \neq b\). Denote \(bab^{-1}\) by \(d\). We have \(ba = db\) and \(b \neq 1\). By G2.6 there exists \(f\) such that \(fa = af\) and \(fd = bf\). Let \(c = fb^{-1}\); then \(c^2 = 1, c \neq 1\), and \(ab = bd\). Then \(fab^{-1} = fbd^{-1}\); but we have \(fab^{-1} = fbf^{-1} fbd^{-1} = ac\) and \(fbd^{-1} = fbf^{-1} fd^{-1} = cb\), so \(ac = cb\).

Lemma 2.3. We have

\[(a^2 = b^2 = 1 \neq a, b \land fa = af \land fb = bf) \rightarrow (a = b \lor f = 1).\]

Proof. Let \(a \neq b\). Assume that \(f \neq 1\). By \(c\) we denote \(fb^{-1}\). By G2.6 there exists \(g\) such that \(gc = cg\) and \(gb = ag\). From the assumption \(bf = fb\) it follows that \(b = c\); so \(gc = gb = bg\), which means that \(ag = bg\) and, finally, \(a = b\).

Given a group satisfying axioms G2.1–G2.8 we may construct an affine plane.

Let \(G = \langle G, 1, \cdot \rangle\) be a group.

Definition 2.1.

\[S(G) := \text{Inv}(G) = \{a \in G: a^2 = 1 \neq a\},\]

\[ab \parallel_G cd :\Leftrightarrow \exists f (f \in G \land fa = cf \land fb = df) \lor a = b \lor c = d\]

\[\land a, b, c, d \in S(G),\]

\[A(G) := \langle S(G), \parallel_G \rangle.\]

Theorem 2.2. Let \(M \models A_{f_2} + F + P\). Then \(M \cong A(\text{Dil}(M))\).

Proof. From Theorems 1.4 and 1.5 it follows that \(S(\text{Dil}(M)) = \{\sigma_p: p \in |M|\}\) and \(\sigma\) transforms \(|M|\) one-to-one onto \(S(\text{Dil}(M))\). From Theorems 1.3 and 1.6 it follows that \(\sigma\) preserves parallelity, which means that \(\sigma\) is an isomorphism.

Theorem 2.3. If \(G \models \text{Dil}_2\), then \(A(G) \models A_{f_2}\).

Proof. Take \(G = \langle G, 1, \cdot \rangle\) such that \(G \models \text{Dil}_2\). Let \(S = S(G)\) and \(\parallel = \parallel_G\). We shall check if in \(\langle S, \parallel \rangle\) axioms S1.0–S1.6 are true:

1° \(ab \parallel ba, a, b \in S\). By Lemma 2.2 there exists \(c \in S\) such that \(ac = cb\).

2° \(a \neq b \land ab \parallel pq \land ab \parallel rs \rightarrow pq \parallel rs\). Let \(a, b, p, q, r, s \in S\). If \(p = q\) or \(r = s\), then \(pq \parallel rs\) from 1°. If \(p \neq q\) and \(r \neq s\), then there exist \(f\) and \(g\) such that
\[fa = pf, \quad fb = qf \quad \text{and} \quad ga = rg, \quad gb = sg. \quad \text{Therefore,} \quad gf^{-1}p = rfg^{-1} \quad \text{and} \quad gf^{-1}q = sgf^{-1}.\]

3° \(ab \parallel ac \rightarrow ba \parallel bc.\) Let \(a, b, c \in S\) and \(ab \parallel ac.\) If \(a = b\) or \(b = c\) or \(a = c,\) then \(ba \parallel bc.\) If \(a \neq b, b \neq c, a \neq c,\) then there is \(f\) such that \(fa = af\) and \(fb = cf, f \neq 1.\) By G2.6 there exists \(g\) such that \(gb = bg\) and \(ga = cg.\)

4° \((\exists a, b, c) \sim ab \parallel ac.\) By G2.7 there exist \(a, b, c \in S\) such that \(a \neq b, c\) and, for any \(d, ad = da\) implies \(db \neq cd.\) This gives \(\sim ab \parallel ac.\)

5° \((\forall a, b, p) \exists q (ab \parallel pq \land p \neq q).\) Let \(a, b, p \in S.\) If \(a = b,\) then choose arbitrary \(q \in S, q \neq p.\) If \(a \neq b,\) take \(r \in S\) such that \(rp = br\) and put \(q = rar.\)

Remark. We have \(rp = br\) and \(ra = qr;\) in other words, \(pa \parallel bq.\) Therefore, \(q\) and \(a, b, p\) form a parallelogram. Moreover, we do not need G2.8 to prove the existence of \(q.\)

6° \(\sim ab \parallel cd \rightarrow \exists p (pa \parallel pb \land pc \parallel pd).\) Let \(a, b, c, d \in S\) and \(\sim ab \parallel cd;\) then \(a \neq b\) and \(c \neq d.\) From 5° we have \(q \neq 0\) such that \(ba \parallel cq.\) So \(\sim cq \parallel cd.\) Now we choose \(p\) in accordance with axiom G2.8.

THEOREM 2.4. If \(G \models D1_2,\) then \(G \equiv \text{Dil}(A(G)).\)

Proof. Let \(S = S(G)\) and let \(\lambda : G \rightarrow S^S\) be defined as follows: for each \(a \in S, \lambda_g(a) = gag^{-1};\) in other words, \(\lambda\) is the conjugation corresponding to \(g,\) restricted to \(S.\) Thus \(\lambda\) preserves the group operation and is well defined, i.e., \(a \in S\) implies \(\lambda_g(a) \in S\) for every \(g \in G.\)

1° \(\lambda\) is one-to-one. Take \(g, h \in G\) and let \(\lambda_g = \lambda_h.\) Pick \(a, b \in S\) such that \(a \neq b.\) Then \(\lambda_g(a) = \lambda_h(a)\) and \(\lambda_g(b) = \lambda_h(b);\) in other words, \(h^{-1}gag^{-1} = a\) and \(h^{-1}gbg^{-1} = h = b,\) so \(g = h\) by Lemma 2.3.

2° For each \(g \in G, \lambda_g \in \text{Dil}(A(G)).\) This fact follows from the definitions of \(\parallel\) and \(\lambda.\)

3° \(\lambda\) transforms \(G\) onto \(\text{Dil}(A(G)).\) Let us take \(a, b \in S, a \neq b,\) and \(\gamma \in \text{Dil}(A(G)).\) We have \(ab \parallel \gamma(a) \gamma(b)\) and \(\gamma(a) \neq \gamma(b).\) Therefore, there exists \(g\) such that \(ga = \gamma(a)g\) and \(gb = \gamma(b)g.\) Thus \(\gamma(a) = \lambda_g(a)\) and \(\gamma(b) = \lambda_g(b).\)

Now the equality \(\gamma = \lambda_g\) is a consequence of Theorem 1.2 and 2°.

THEOREM 2.5. \(G \models D1_2 \rightarrow A(G) \models F.\)

Proof. Take \(a, b, c, d \in S(G)\) such that \(ad \parallel cb, ac \parallel bd\) and let \(\sim ac \parallel cb.\) Pick \(p \in S(G)\) such that \(pa = bp.\) Denote \(pd^{-1}\) by \(c'.\) We have \(pd = c'p, ad \parallel c'b,\) and \(ac' \parallel bd.\) From axiom S1.2 we obtain \(cb \parallel c'b\) and \(ac \parallel ac',\) and from S1.3 we get \(cc' \parallel cb\) and \(cc' \parallel ac.\) If \(c \neq c',\) then we would have \(cb \parallel ac.\) Hence \(c = c'\) because \(\sim cb \parallel ac.\) Thus we obtain \(ap \parallel ab\) and \(cp \parallel cd.\)

THEOREM 2.6. \(G \models D1_2 \rightarrow A(G) \models D.\)

Proof. On the affine plane the sentence \(D\) holds if and only if the groups of homotheties with fixed center are transitive. Let \(a, b, c \in S, a \neq b, c,\) and \(ab \parallel ac.\) By the definition of parallelity there exists \(f\) such that \(fa = af\) and \(fb = cf.\) We take \(\lambda_f\) for the dilatation we look for.

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Theorem 2.7. $G \models \text{Di}_2 \rightarrow A(G) \models P$.

Proof. On the Desarguesian affine plane the sentence $P$ holds if and only if the groups of homotheties with fixed center are abelian.

Let $\beta, \gamma \in J_a(A(G))$, where $a \in S$. There exist $h$ and $k$ such that $\beta = \lambda_h$ and $\gamma = \lambda_k$. We have $hah^{-1} = a$ and $kak^{-1} = a$. Therefore $ha = ah$ and $ka = ak$.

By axiom G2.5 we have $hk = kh$, so $\beta \gamma = \gamma \beta$.

Theorem 2.8 (representation). We have

$$G \models \text{Di}_2 \iff \exists M \ (M \models A_f + P + F \land G \cong \text{Dil}(M)).$$

Proof. $\rightarrow$ is a direct consequence of Theorems 2.3–2.5 and 2.7 (for $M$ take $A(G)$).

$\leftarrow$ is a consequence of Theorem 2.1.

Below we give a new "group" theorem about representation for 2-dimensional affine geometry.

Theorem 2.9. $M \models A_f + P + F \iff \exists G (G \models \text{Di}_2 \land M \cong A(G))$.

Proof. $\rightarrow$ is a consequence of Theorems 2.1 and 2.2 (for $G$ take $\text{Dil}(M)$).

$\leftarrow$ is a consequence of Theorems 2.3, 2.5 and 2.7.

3. Remarks on the axiom system Di$_2$. Now we show to which geometrical facts some of the axioms of Di$_2$ correspond. Let Di$_2^P$ be the theory based upon axioms G2.1–G2.4 and G2.6–G2.8.

Theorem 3.1. $G \models \text{Di}_2^P \iff \exists M \ (M \models A_f + D + F \land G \cong \text{Dil}(M))$.

Proof. It suffices to notice that if $G \models \text{Di}_2^P$, then

$$A(G) \models A_f + D + F \quad \text{and} \quad G \cong \text{Dil}(A(G))$$

(in the proofs of Theorems 2.3–2.6 we did not use G2.5).

If $M \models A_f + D + F$, then $\text{Dil}(M) \models \text{Di}_2^P$.

Corollary 3.1. If $G \models \text{Di}_2^P$, then $G \models G2.5 \iff A(G) \models P$.

Axiom G2.5 is therefore equivalent to the axiom of Pappus.

Lemma 3.1. If $G \models G2.1$–G2.7, then

$$A(G) \models S1.1$–S1.5 + P + F \quad \text{and} \quad G \cong \text{Dil}(A(G)).$$

Proof. Notice that in the proofs of Theorems 2.3 (0°–5°) and 2.4–2.7 axiom G2.8 was not used.

Remark 3.1. In the proof of Theorem 2.4 the rigidity of the plane dilatation group was used, but this fact holds for dimension-free affine structures satisfying S1.1–S1.5 only.

Theorem 3.2. If $G \models G2.1$–G2.7, then

$$G \models G2.8 \iff A(G) \models S1.6.$$
From the assumption and Lemma 3.1 we obtain $A(G) \models A_{f_2}$ and $G \cong \text{Dil}(A(G))$. By Theorem 2.1, $\text{Dil}(A(G)) \models G_{2.8}$.

We can say that axiom $G_{2.8}$ is equivalent to the upper axiom of dimension.

Let $A_{f_{\geq 3}}$ be a theory of affine spaces of dimension 3 or higher based upon Kusak's axioms, i.e., S1.1–S1.3, S1.6,

\[ \exists d \ (ab \parallel cd \land ac \parallel bd), \sim ab \parallel ac \land ad \parallel ac \rightarrow \exists p \ (ab \parallel dp \land bc \parallel bp). \]

**Lemma 3.2.** If $G \models G_{2.1}–G_{2.7}$, then

(i) $A(G) \models \exists d \ (ab \parallel cd \land ac \parallel bd)$,

(ii) $A(G) \models \sim ab \parallel ac \land ad \parallel ac \rightarrow \exists p \ (ab \parallel dp \land bc \parallel bp)$.

**Proof.** (i) has been already shown in the proof of Theorem 2.3 (see the Remark to 5°).

(ii) Take $a, b, c, d \in S(G)$. From the assumption that $ab \parallel ac$ we obtain $ca \parallel cd$. By the assumption $\sim ab \parallel ac$ we have $a \neq c$; then there exists an $f$ such that $fc = cf$ and $fa = df$. Take $p = fbf^{-1}$; in other words, $fb = pf$. Thus $ab \parallel dp$ and $cb \parallel cp$, whence $ab \parallel dp$ and $bc \parallel bp$. (If $c = d$, then $p = c$.)

**Theorem 3.3.** We have

$G \models G_{2.1}–G_{2.7}, \sim G_{2.8} \Leftrightarrow \exists M \ (G \cong \text{Dil}(M) \land M \models A_{f_{\geq 3}} + F + P)$.

This theorem is a consequence of Theorem 3.2 and Lemmas 3.1 and 3.2.

**Corollary 3.2.** We have

$G \models G_{2.1}–G_{2.7}$

$\Leftrightarrow \exists M \ (G \cong \text{Dil}(M) \land (M \models A_{f_2} + P + F \lor M \models A_{f_{\geq 3}} + P + F))$.

**Remark 3.2.** In $A_{f_{\geq 3}}$, the sentence $D$ can be proved (see [1]).

**Remark 3.3.** In $A_{f_{\geq 3}} + F$, axiom $G_{2.5}$ is equivalent to the sentence $P$.

**Remark 3.4.** In $\text{Dil}_2^2$, axiom $G_{2.8}$ is also equivalent to the upper axiom of dimension.

**References**


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